CUK-H03275-13-P014725

510.5
.002

AMERICAN

Journal of Mathematics.

⑬

SIMON NEWCOMB, Editor.
THOMAS CRAIG, Associate Editor.

PUBLISHED UNDER THE AUSPICES OF THE

JOHNS HOPKINS UNIVERSITY.

Πραγμάτων ἔλεγχος οὐ βλεπομένων.

UNIVERSITY OF CALCUTTA
LIBRARY.
THE ADVANCEMENT OF LEARNING.

VOLUME XIV.

BALTIMORE: THE JOHNS HOPKINS PRESS.

AGENTS:

B. WESTERMANN & Co., *New York.*
D. VAN NOSTRAND Co., *New York.*
E. STEIGER & Co., *New York.*
G. E. STECHERT, *New York.*
A. C. McCLURG & Co., *Chicago.*
KEGAN PAUL, TRENCH, TRÜBNER & Co., *London.*
A. HERMANN, *Paris.*
GAUTHIER-VILLARS, *Paris.*
MAYER & MÜLLER, *Berlin.*
KARL J. TRÜBNER, *Strassburg.*

1892.

D16725

PRESS OF THE FRIEDENWALD CO.,
BALTIMORE, MD.

UNIVERSITY OF CALCUTTA
LIBRARY.
THE ADVANCEMENT OF LEARNING.

INDEX

PAGE

APPELL, P. Sur une expression nouvelle des fonctions elliptiques par le quotient de deux séries, ... 9
BROWN, ERNEST W. On the Part of the Parallactic Inequalities in the Moon's Motion which is a Function of the Mean Motions of the Sun and Moon, ... 141
CAYLEY, A. Corrected Seminvariant Tables for the Weights 11 and 12, ... 195
CHAPMAN, C. H. Application of Quaternions to Projective Geometry, ... 115
COLE, F. N. Simple Groups from Order 201 to Order 500, ... 378
FIELDS, J. C. Transformation of a System of Independent Variables, ... 230
GOURSAT, E. Sur un problème relatif à la déformation des surfaces, ... 1
HARRIS, ROLLIN A. Note on the Use of Supplementary Curves in Isogonal Transformation, ... 291
HULBURT, L. S. A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves, ... 246
JOHNSON, WM. WOOLSEY. Some Theorems relating to Groups of Circles and Spheres, 97
McCLINTOCK, EMORY. On the Algebraic Proof of a Certain Series, ... 67
  On Independent Definitions of the Functions log (x) and e^x, ... 72
  On the Computation of Covariants by Transvection, ... 222
MACMAHON, P. A. Fourth Memoir on a New Theory of Symmetric Functions, . 15
MANNING, H. P. A Note on Linear Transformation, ... 95
MERRIMAN, MANSFIELD. The Deduction of Final Formulas for the Algebraic Solution of the Quartic Equation, ... 237
METZLER, W. H. On the Roots of Matrices, ... 326
NEWSON, H. B. A Pair of Curves of the Fourth Degree and their Application in the Theory of Quadrics, ... 87
D'OCAGNE, M. Extrait d'une lettre à M. Craig, ... 389
OSGOOD, WILLIAM F. The Symbolic Notation of Aronhold and Clebsch, ... 251
  The System of Two Simultaneous Ternary Quadratic Forms, ... 262
POINCARÉ, H. Sur les fonctions à espaces lacunaires, ... 201
SCOTT, CHARLOTTE ANGAS. On the Higher Singularities of Plane Curves, . . . 301
STEINMETZ, CHAS. PROTEUS. On the Curves which are Self-reciprocal in a Linear Nulsystem, and their Configurations in Space, ... 161
  Multivalent and Univalent Involutory Correspondences in a Plane determined by a Net of Curves of n^{th} Order, ... 39
STRINGHAM, IRVING. A Classification of Logarithmic Systems, ... 187
WHITE, HENRY S. On Generating Systems of Ternary and Quaternary Linear Transformations, ... 274
  A Symbolic Demonstration of Hilbert's Method for deriving Invariants and Covariants of given Ternary Forms, ... 283

Felix Klein.

Sur un Problème relatif à la déformation des surfaces.

Par E. Goursat.

1. Dans son Mémoire sur la déformation des surfaces,[*] Bour a montré qu'on pouvait toujours, étant donnée une surface de révolution quelconque, trouver une infinité d'autres surfaces de révolution applicables sur la première, de telle façon que les méridiens et les parallèles se correspondent. Une propriété analogue appartient encore aux surfaces moulures et à une classe plus générale de surfaces que nous allons définir. Nous nous proposons pour cela le problème général suivant:[†]

Peut-on déformer une surface de telle façon qu'une série de sections planes, dont les plans sont parallèles, se change en une série de sections planes dont les plans sont parallèles?

Soit $z = f(x, y)$ l'équation d'une surface S. Toute surface S' applicable sur la première peut être définie en regardant les coordonnées X, Y, Z d'un point de S' comme des fonctions de x, y, assujetties à vérifier la relation

$$dX^2 + dY^2 + dZ^2 = (1 + p^2)\,dx^2 + 2pq\,dx\,dy + (1 + q^2)\,dy^2, \tag{1}$$

où

$$p = \frac{\partial f}{\partial x}, \quad q = \frac{\partial f}{\partial y}.$$

Chacune des fonctions X, Y, Z doit vérifier l'équation aux dérivées partielles du second ordre[‡]

$$(s^2 - rt)(P^2 + Q^2 - 1) + (S^2 - RT)(1 + p^2 + q^2) + (rT + tR + 2sS)(P^p + Q^q) = 0, \tag{2}$$

---

[*] Journal de l'Ecole Polytechnique, 39ème Cahier.

[†] Depuis que ces lignes sont écrites, j'ai eu connaissance d'un travail de Mr. B. Mlodzieiowski, publié dans le Bulletin des Sciences Mathématiques (Avril 1891), où l'auteur signale en passant les surfaces étudiées ici. Mais il ne s'est pas posé la question à un point de vue général et n'a pas indiqué le mode de génération de ces surfaces. E. G.

[‡] Darboux: Leçons sur la théorie générale des surfaces, t. III, p. 262.

P, Q, R, S, T désignant les dérivées premières et secondes de la fonction inconnue. On ne restreint évidemment pas la généralité du problème en supposant que ce sont les sections planes parallèles au plan des yz qui se correspondent dans les deux surfaces, de façon que X se réduise à une fonction de x seulement

$$X = \phi(x).$$

L'équation (2) devra admettre pour intégrale une fonction de x seulement. Pour cette intégrale on aura $Q = 0$, $S = 0$, $T = 0$, et l'équation (2) se réduit à

$$(s^2 - rt)(P^2 - 1) + tpRP = 0.$$

On a toujours la solution évidente $P^2 = 1$, ou

$$X = \pm x + \varkappa$$

qui ne fournit que des surfaces S' égales à S ou à sa symétrique. Pour qu'il existe d'autres solutions que celle-là, il faut et il suffit que

$$\frac{s^2 - rt}{tp}$$

soit une fonction de x seulement

$$s^2 - rt = tp\,F(x), \tag{3}$$

et le problème se ramène à l'intégration de l'équation (3). Nous voyons déjà que les solutions connues ne peuvent être les plus générales, puisque l'équation (3) contient une fonction arbitraire $F(x)$, et l'intégration en introduira deux autres. On voit en outre que, si la surface S satisfait à l'équation (3), il existera une infinité de surfaces S' répondant à la question et dépendant d'une constante arbitraire, car on aura pour déterminer $X = \phi(x)$ une équation différentielle du second ordre

$$F(x)\{\phi'^2 - 1\} + \phi'\phi'' = 0,$$

dont l'intégrale générale est

$$\phi = \int \sqrt{1 + Ce^{-2\int F(x)\,dx}}\,dx + C'.$$

On peut toujours supposer $C' = 0$, car cela revient à transporter la surface parallèlement à elle-même.

2. L'équation (3) s'intègre facilement par la méthode de Monge et d'Ampère. On aura pour les équations différentielles des caractéristiques*

$$\begin{aligned} dpdx + dqdy + p\,F(x)\,dx^2 &= 0, \\ dpdq + p\,F(x)\,dqdx &= 0; \end{aligned}$$

---

* Darboux: t. III, p. 264.

on en déduit aussitôt les deux combinaisons intégrables

$$dq = 0, \quad dp + p\,F(x)\,dx = 0.$$

Comme $F(x)$ est une fonction arbitraire de x, on peut toujours poser

$$F(x) = \frac{\psi'(x)}{\psi(x)},$$

$\psi(x)$ désignant encore une fonction arbitraire. On obtient alors pour l'équation (3) l'intégrale intermédiaire

$$p\psi(x) = \phi(q), \tag{4}$$

$\phi(q)$ étant une fonction arbitraire. Pour intégrer cette équation par la méthode de Lagrange et Charpit, je pose $q = a$, a étant une constante arbitraire; on aura ensuite

$$p = \frac{\phi(a)}{\psi(x)},$$

$$dz = \frac{\phi(a)}{\psi(x)}\,dx + a\,dy,$$

ce qui fournit l'intégrale complète

$$z = \phi(a) f(x) + ay + b, \tag{5}$$

où

$$f(x) = \int \frac{dx}{\psi(x)}.$$

La méthode de la variation des constantes donnera ensuite l'intégrale générale et, en changeant un peu les notations, on a les formules définitives

$$\left.\begin{aligned} y &= f(x)\,\phi'(a) + \psi'(a), \\ z &= f(x)[\phi(a) - a\phi'(a)] + \psi(a) - a\psi'(a), \end{aligned}\right\} \tag{6}$$

qui donnent l'intégrale générale de l'équation

$$(s^2 - rt) f'(x) + ptf''(x) = 0. \tag*{(3)$^{\text{bis}}$}$$

3. Les surfaces représentées par les équations (6) dépendent des trois fonctions arbitraires f, ϕ, ψ. Ces surfaces sont susceptibles d'une définition géométrique analogue à celle des surfaces moulures. Remarquons d'abord que les courbes $x = C^{\text{t}}$, $a = C^{\text{t}}$ forment sur la surface un réseau conjugué. On a, en effet, en regardant x et a comme les deux variables indépendantes,

$$\left.\begin{aligned} &\frac{\partial x}{\partial x} = 1, && \frac{\partial y}{\partial x} = f'(x)\,\phi'(a), && \frac{\partial z}{\partial x} = f'(x)[\phi(a) - a\phi'(a)], \\ &\frac{\partial x}{\partial a} = 0, && \frac{\partial y}{\partial a} = f(x)\phi''(a) + \psi''(a), && \frac{\partial z}{\partial a} = -a[f(x)\,\phi''(a) + \psi''(a)], \\ &\frac{\partial^2 x}{\partial a \partial x} = 0, && \frac{\partial^2 y}{\partial a \partial x} = f'(x)\,\phi''(a), && \frac{\partial^2 z}{\partial a \partial x} = -af'(x)\,\phi''(a), \end{aligned}\right\} \tag{7}$$

et, par suite,

$$\begin{vmatrix} \dfrac{\partial y}{\partial x} & \dfrac{\partial z}{\partial x} & 1 \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a} & 0 \\ \dfrac{\partial^2 y}{\partial a \partial x} & \dfrac{\partial^2 z}{\partial a \partial x} & 0 \end{vmatrix} = 0.$$

Les courbes $x = C^{\text{t}}$ sont les sections de la surface par des plans parallèles au plan des yz; par conséquent, les courbes $a = C^{\text{t}}$ sont les courbes de contact des cylindres circonscrits ayant leurs génératrices parallèles au plan des yz. Or ces courbes $a = C^{\text{t}}$ sont des courbes planes dont le plan est parallèle à l'axe des x, comme on le voit immédiatement d'après les équations (6). Il suit de là que, si on considère les sections de la surface par des plans parallèles au plan des yz, les tangentes à ces différentes sections aux points où elles sont rencontrées par une même courbe $a = C^{\text{t}}$ sont parallèles. Si on projette toutes ces sections planes sur le plan des yz, on aura un réseau de courbes planes jouissant de la propriété suivante: les points de ces courbes où les tangentes sont parallèles sont toujours sur une même ligne droite. On en déduit la construction suivante de la surface:

Prenons dans le plan des yz deux courbes *quelconques* C, C' et faisons correspondre les points M, M' des deux courbes où les tangentes sont parallèles. Joignons M et M' et prenons le point m qui divise le segment MM' dans un rapport donné $\varkappa$. Lorsque les points M, M' décrivent les courbes C, C', le point m décrit une certaine courbe $C_\varkappa$, et, en faisant varier le rapport $\varkappa$, on forme un réseau de courbes planes. Déplaçons chacune de ces courbes d'une quantité arbitraire parallèlement à Ox; la surface ainsi engendrée est la surface la plus générale répondant à la question. Les deux courbes C, C' et la loi du déplacement donnent bien trois fonctions arbitraires. Parmi les cas particuliers remarquables, on peut citer les suivants:

1°. Si les courbes C, C' sont des circonférences concentriques, la surface obtenue est une surface de révolution. Plus généralement, si les courbes C et C' sont deux courbes *parallèles* quelconques, la construction précédente donne une surface moulure. On aura dans ce cas

$$\phi(a) = \sqrt{a^2 + 1},$$

et, de plus, $\psi(a) = 0$, si la surface est de révolution.

2°. Prenons pour C et C' deux courbes égales, C' se déduisant de C par une simple translation. Alors toutes les courbes C_κ seront égales à C et la surface sera engendrée par un mouvement de translation de la courbe C, chacun des points de cette courbe décrivant une courbe plane dont le plan est parallèle à l'axe des x. Si on suppose en outre que ce plan est parallèle au plan des xz, $\phi(a)$ se réduira à une constante, et la surface aura une équation de la forme

$$z = F(x) + F_1(y);$$

on voit qu'elle admet un double mode de génération par un profil animé d'un mouvement de translation.

3°. Supposons que les courbes C et C' soient homothétiques; les courbes C_κ seront également homothétiques aux deux premières. Si en particulier C et C' sont des ellipses homothétiques et concentriques, la construction donnera, en choisissant convenablement la loi de déplacement, les surfaces du second degré. Il faudra prendre ici $\psi(a) = 0$.

4°. Si $f(x) = Ax + B$, la surface (6) est une surface développable, ce dont il est aisé de se rendre compte par quelques considérations géométriques.

4. Les fonctions f, ϕ, ψ étant connues, proposons-nous maintenant de trouver toutes les surfaces S' applicables sur la surface S, de telle façon que les sections planes parallèles au plan des yz se correspondent. Il est clair que ces nouvelles surfaces seront de même nature que la première. Pour simplifier les formules, nous écrirons les formules (6), en changeant un peu les notations,

$$\left.\begin{aligned} y &= f(x)\,\theta_1(a) + \eta_1(a), \\ z &= f(x)\,\theta(a) + \eta(a), \end{aligned}\right\} \qquad (7)^{\text{bis}}$$

les fonctions θ, θ_1, η, η_1 satisfaisant à la relation

$$\frac{\eta_1'(a)}{\eta'(a)} = \frac{\theta_1'(a)}{\theta'(a)} = \lambda. \qquad (8)$$

On en tire :

$$dy = f'(x)\,\theta_1(a)\,dx + [f(x)\,\theta_1'(a) + \eta_1'(a)]\,da,$$
$$dz = f'(x)\,\theta(a)\,dx + [f(x)\,\theta'(a) + \eta'(a)]\,da,$$
$$\left.\begin{aligned} ds^2 = dx^2 + dy^2 + dz^2 &= dx^2 + f'^2(x)[\theta^2(a) + \theta_1^2(a)]\,dx^2, \\ &+ 2f'(x)[f(x)\{\theta_1(a)\,\theta_1'(a) + \theta(a)\,\theta'(a)\} + \theta_1(a)\,\eta_1'(a) + \theta(a)\,\eta'(a)]\,da\,dx \\ &+ [f^2(x)\{\theta'^2(a) + \theta_1'^2(a)\} + 2f(x)\{\theta'(a)\,\eta'(a) + \theta_1'(a)\,\eta_1'(a)\} \\ &\qquad + \eta'^2(a) + \eta_1'^2(a)]\,da^2. \end{aligned}\right\} \qquad (9)$$

Soient X, Y, Z les coordonnées d'un point de la nouvelle surface, que nous supposerons exprimées en fonction de x et de a; X étant supposée fonction de x seulement, on aura, nous l'avons vu,

$$\frac{X'X''}{X'^2-1}=\frac{rt-s^2}{tp}=\frac{f''(x)}{f'(x)},$$

et on en déduit, en négligeant une constante additive,

$$X=\int\sqrt{1+nf'^2(x)}\,dx, \tag{10}$$

n désignant un paramètre arbitraire. Nous prendrons pour Y et Z des expressions de la forme suivante

$$\left.\begin{aligned}Y&=f(x)\,\Theta_1(a)+\mathrm{H}_1(a),\\ Z&=f(x)\,\Theta(a)\;+\mathrm{H}(a),\end{aligned}\right\} \tag{11}$$

$\Theta(a)$, $\Theta_1(a)$, $\mathrm{H}(a)$, $\mathrm{H}_1(a)$ étant des fonctions à déterminer. Des formules (10) et (11) on déduit, pour le carré de l'élément linéaire de la nouvelle surface,

$$\left.\begin{aligned}dS^2=dx^2&+f'^2(x)[\Theta^2(a)+\Theta_1^2(a)+n]\,dx^2\\ &+2f'(x)[f(x)\{\Theta_1(a)\,\Theta_1'(a)+\Theta(a)\,\Theta'(a)\}\\ &\qquad\qquad+\Theta_1(a)\,\mathrm{H}_1'(a)+\Theta(a)\,\mathrm{H}'(a)]\,da\,dx\\ &+[f'^2(x)\{\Theta'^2(a)+\Theta_1'^2(a)\}+2f(x)\{\Theta'(a)\,\mathrm{H}'(a)+\Theta_1'(a)\,\mathrm{H}_1'(a)\}\\ &\qquad\qquad+\mathrm{H}'^2(a)+\mathrm{H}_1'^2(a)]\,da^2.\end{aligned}\right\} \tag{12}$$

En identifiant les formules (9) et (12), on obtient les relations ci-dessous:

$$\begin{aligned}&(\mathrm{A})\quad \Theta^2(a)+\Theta_1^2(a)=\theta^2(a)+\theta_1^2(a)-n,\\ &(\mathrm{B})\quad \Theta(a)\,\Theta'(a)+\Theta_1(a)\,\Theta_1'(a)=\theta(a)\,\theta'(a)+\theta_1(a)\,\theta_1'(a),\\ &(\mathrm{C})\quad \Theta(a)\,\mathrm{H}'(a)+\Theta_1(a)\,\mathrm{H}_1'(a)=\theta(a)\,\eta'(a)+\theta_1(a)\,\eta_1'(a),\\ &(\mathrm{D})\quad \Theta'^2(a)+\Theta_1'^2(a)=\theta'^2(a)+\theta_1'^2(a),\\ &(\mathrm{E})\quad \Theta'(a)\,\mathrm{H}'(a)+\Theta_1'(a)\,\mathrm{H}_1'(a)=\theta'(a)\,\eta'(a)+\theta_1'(a)\,\eta_1'(a),\\ &(\mathrm{F})\quad \mathrm{H}'^2(a)+\mathrm{H}_1'^2(a)=\eta'^2(a)+\eta_1'^2(a).\end{aligned}$$

La relation (B) est une conséquence de la relation (A); les deux équations (A) et (D) déterminent $\Theta(a)$ et $\Theta_1(a)$ par une quadrature. On en déduit, en effet,

$$\begin{aligned}&(\Theta^2(a)+\Theta_1^2(a))(\Theta'^2(a)+\Theta_1'^2(a))\\ &\qquad=(\Theta(a)\,\Theta'(a)+\Theta_1(a)\,\Theta_1'(a))^2+\{\Theta(a)\,\Theta_1'(a)-\Theta_1(a)\,\Theta'(a)\}^2,\end{aligned}$$

c. à d.

$$\begin{aligned}&\{\theta^2(a)+\theta_1^2(a)-n\}\{\theta'^2(a)+\theta_1'^2(a)\}\\ &\qquad=\{\theta(a)\,\theta'(a)+\theta_1(a)\,\theta_1'(a)\}^2+\{\Theta(a)\,\Theta_1'(a)-\Theta_1(a)\,\Theta'(a)\}^2,\end{aligned}$$

et par suite

$$\frac{\Theta(a)\,\Theta_1'(a)-\Theta_1(a)\,\Theta'(a)}{\Theta^2(a)+\Theta_1^2(a)}=\frac{d}{da}\operatorname{arc\,tg}\frac{\Theta_1(a)}{\Theta(a)}$$

$$=\frac{\sqrt{\{\theta^2+\theta_1^2-n\}\{\theta'^2+\theta_1'^2\}-\{\theta\theta'+\theta_1\theta_1'\}^2}}{\theta^2+\theta_1^2-n}.$$

Une fois $\Theta(a)$ et $\Theta_1(a)$ obtenus, on aura 3 équations (C), (E), (F) pour calculer $H'(a)$, $H_1'(a)$. Les équations (C) et (E) étant du premier degré, on en tire

$$H'=\frac{\Theta_1'\{\theta\eta'+\theta_1\eta_1'\}-\Theta_1\{\theta'\eta'+\theta_1'\eta_1'\}}{\Theta\Theta_1'-\Theta_1\Theta'},$$

$$H_1'=\frac{\Theta\{\theta'\eta'+\theta_1'\eta_1'\}-\Theta'\{\theta\eta'+\theta_1\eta_1'\}}{\Theta\Theta_1'-\Theta_1\Theta'}.$$

En tenant compte des relations (8), (A), (B), (D), on vérifie sans peine que ces valeurs de H', H_1' vérifient la dernière équation (F).

5. Les formules précédentes renferment, comme cas particuliers, les formules de Bour. Si on suppose $\phi(a)=1$, on est conduit au résultat suivant, dont la vérification est immédiate: toutes les surfaces représentées par les formules

$$\left.\begin{aligned}X&=\int\sqrt{1+(1-n^2)F'^2(x)}\,dx,\\ Y&=\int\sqrt{1+\left(1-\frac{1}{n^2}\right)F_1'^2(y)}\,dy,\\ Z&=nF(x)+\frac{F_1(y)}{n}\end{aligned}\right\}$$

sont applicables sur la surface qui a pour équation

$$z=F(x)+F_1(y).$$

Si on suppose $\eta=\eta_1=0$, on aura aussi $H'=H_1'=0$, et on n'a plus que deux quadratures à effectuer, l'une pour avoir X, l'autre pour avoir $\operatorname{arc\,tg}\frac{\Theta_1}{\Theta}$. Remarquons que ces deux quadratures sont absolument indépendantes l'une de l'autre.

Prenons par exemple l'ellipsoïde

$$\left.\begin{aligned}y&=\sqrt{1-\frac{x^2}{A^2}}\,B\cos t,\\ z&=\sqrt{1-\frac{x^2}{A^2}}\,C\sin t,\end{aligned}\right\}$$

où A, B, C sont les trois demi-axes, x et t les paramètres variables. Posons $\Theta^2(t)+\Theta_1^2(t)=B^2\cos^2 t+C^2\sin^2 t-n$; on aura ensuite,

$$\frac{d}{dt}\operatorname{arc\,tg}\frac{\Theta_1(t)}{\Theta(t)}$$

$$=\frac{\sqrt{\{(B^2-n)\cos^2 t+(C^2-n)\sin^2 t\}\{B^2\sin^2 t+C^2\cos^2 t\}-(B^2-C^2)^2\sin^2 t\cos^2 t}}{(B^2-n)\cos^2 t+(C^2-n)\sin^2 t}$$

toutes les surfaces représentées par le système des 3 équations

$$\left.\begin{aligned} X&=\int\sqrt{1+\frac{nx^2}{A^4\left(1-\frac{x^2}{A^2}\right)}}\,dx, \\ Y&=\sqrt{1-\frac{x^2}{A^2}}\,\Theta(t), \\ Z&=\sqrt{1-\frac{x^2}{A^2}}\,\Theta_2(t) \end{aligned}\right\}$$

sont applicables sur l'ellipsoïde. Si on pose $\operatorname{tg} t=u$, il vient:

$$\operatorname{arc\,tg}\frac{\Theta_1}{\Theta}=\int\frac{\sqrt{\{B^2-n+(C^2-n)u^2\}\{B^2u^2+C^2\}-(B^2-C^2)^2u^2}}{B^2-n+(C^2-n)u^2}\times\frac{du}{1+u^2};$$

on est ramené à deux intégrales elliptiques de modules différents.

On trouverait de même des surfaces applicables sur le paraboloïde quelconque, mais les résultats obtenus plus haut fournissent une solution encore plus simple. Ainsi les surfaces représentées par les trois équations

$$\left.\begin{aligned} X&=\int\sqrt{1+(1-n^2)\frac{x^2}{a^2}}\,dx, \\ Y&=\int\sqrt{1+\left(1-\frac{1}{n^2}\right)\frac{y^2}{b^2}}\,dy, \\ Z&=\frac{nx^2}{2a}+\frac{y^2}{2bn} \end{aligned}\right\}$$

sont toutes applicables sur le paraboloïde

$$z=\frac{x^2}{2a}+\frac{y^2}{2b}.$$

PARIS, *Mai* 1891.

Sur une expression nouvelle des fonctions elliptiques par le quotient de deux séries.

Par P. Appell.

D'après les théorêmes généraux de la théorie des fonctions, on reconnait *a priori* l'existence d'une infinité de représentations analytiques d'une fonction *méromorphe dans tout le plan*, c'est à dire uniforme et n'ayant que des pôles à distance finie, ces représentations analytiques étant assujetties à donner la fonction pour toutes les valeurs de la variable. Si l'on se limite aux représentations qui donnent la fonction sous forme du quotient de deux séries convergentes pour toutes les valeurs de la variable, il existe encore une infinité de représentations différentes. Les plus simples sont 1°) celle qui donne la fonction sous forme du quotient de deux séries entières, par exemple celle qui donne les fonctions elliptiques sous forme du quotient de fonctions Θ; 2°) celle qui donne la fonction sous forme d'une série unique mettant en évidence les pôles et les parties principales correspondantes, série qui est définie par le théorême de M. Mittag-Leffler.

Plus généralement, en admettant que la fonction ait pour zéros les points

$$a_1, a_2, \ldots . a_\nu, \ldots$$

et pour pôles les points

$$b_1, b_2, \ldots . b_\nu, \ldots ,$$

on pourra la regarder comme le quotient de deux fonctions méromorphes

$$\frac{P(z)}{Q(z)},$$

la fonction $P(z)$ ayant pour zéros une partie des points a_ν et pour pôles une partie des points b_ν; la fonction $Q(z)$ ayant pour zéros les autres points b_ν et pour pôles les autres points a_ν. Ces deux fonctions P et Q sont, d'après le théorême de M. Mittag-Leffler, représentées par des séries convergentes dans tout le plan.

2

La seule difficulté qui se présentera et qui pourra être considérable sera de calculer les coefficients des séries et surtout ceux de la partie entière des développements.

Nous nous proposons ici de faire ce calcul pour les fonctions elliptiques en prenant, pour $Q(z)$, une fonction entière ayant pour zéros les pôles de la fonction situés dans une moitié du plan et, pour $P(z)$, une fonction ayant pour pôles les pôles de la fonction situés dans l'autre moitié du plan.

Ces recherches se rattachent à une étude dejà ancienne* sur les fonctions que Heine a introduites comme une généralisation des fonctions Eulériennes Γ,† et à des résultats que M. Poincaré a indiqués dans ses mémoires sur les invariants arithmétiques (Comptes Rendus 1879 et Congrés de l'Association française pour l'avancement des Sciences, Alger 1881). Elles donnent les fonctions elliptiques sous une forme nouvelle mettant en évidence la double périodicité d'une manière différente de celle qui se présente dans les expressions connues. Voici comment l'on arrive à cette forme.

Soient ω et ω' les deux périodes choisies, comme on le fait habituellement, de telle façon que le module de la quantité

$$q = e^{\frac{\pi\omega' i}{\omega}}$$

soit moindre que l'unité. Posons

$$\left.\begin{aligned} Q_0 &= \prod_1^\infty (1-q^{2n}) = (1-q^2)(1-q^4)(1-q^6)\ldots\ldots \\ Q_1 &= \prod_1^\infty (1+q^{2n}) = (1+q^2)(1+q^4)(1+q^6)\ldots\ldots \end{aligned}\right\} \qquad (1)$$

La fonction entière de z

$$G(z) = \frac{1}{Q_0}\left[1 + \sum_{n=1}^{n=\infty} (-1)_n \frac{q^{n(n+1)}}{(1-q^2)(1-q^4)\ldots\ldots(1-q^{2n})} e^{\frac{2n\pi z i}{\omega}}\right], \qquad (2)$$

vérifie, comme on le voit sans peine, les deux relations

$$\left.\begin{aligned} G(z+\omega) &= G(z), \\ \left(1 - q^2 e^{\frac{2\pi z i}{\omega}}\right) G(z+\omega') &= G(z); \end{aligned}\right\} \qquad (3)$$

---

*Comptes Rendus des Séances de l'Académie des Sciences, T. 89, pages 841 et 1081.—Mathematische Annalen 1881.

† Heine: Handbuch der Kugelfunctionen, p. 109 et suiv.

cette fonction $G(z)$ est identique à la fonction qui, d'après les notations de Heine, s'écrirait

$$\frac{1}{\Omega\left(q^2, \frac{z}{\omega'}\right)}.^{*}$$

Si l'on avait une autre fonction $F(z)$ satisfaisant aux deux relations

$$\left.\begin{aligned} F(z+\omega) &= F(z), \\ \left(1 - q^2 e^{\frac{2\pi z i}{\omega}}\right) F(z+\omega') &= -F(z), \end{aligned}\right\} \tag{4}$$

le quotient $\frac{F(z)}{G(z)}$ fournirait une fonction doublement périodique de z aux périodes ω et $2\omega'$. Cherchons d'abord s'il n'existe pas de série entière en $e^{\frac{2\pi z i}{\omega}}$ de la forme

$$F(z) = \sum_{n=-\infty}^{n=+\infty} A_n e^{\frac{2n\pi z i}{\omega}}$$

convergente dans une certaine bande du plan des z et vérifiant les relations (4). Si, pour simplifier, nous faisons

$$e^{\frac{2\pi z i}{\omega}} = x, \quad q^2 = t$$

la série cherchée $F(z)$ deviendra

$$F_1(x) = \sum_{-\infty}^{+\infty} A_n x^n$$

et elle devra vérifier la relation

$$(tx - 1)\, F_1(tx) = F_1(x). \tag{5}$$

Cette équation donne entre deux coefficients consécutifs la relation

$$A_n (1 + t^n) - A_{n-1} t^n = 0.$$

D'où, en supposant n positif

$$A_n = A_0 \frac{t^{\frac{n(n+1)}{2}}}{(1+t)(1+t^2)\ldots\ldots(1+t^n)}$$

et

$$A_{-n-1} = 2A_0 (1+t)(1+t^2)\ldots\ldots(1+t^n).$$

Par suite, en prenant $A_0 = 1$, et posant

$$\phi(x) = 1 + \sum_{n=1}^{n=\infty} \frac{t^{\frac{n(n+1)}{2}}}{(1+t)(1+t^2)\ldots\ldots(1+t^n)} x^n \tag{6}$$

---

[*] Handbuch der Kugelfunctionen, page 105, éq. (4, b).

et

$$\psi(x) = \frac{2}{x} + 2\sum_{n=1}^{n=\infty} \frac{(1+t)(1+t^2)\ldots\ldots(1+t^n)}{x^{n+1}} \tag{7}$$

on aura, pour satisfaire à l'équation (5) la fonction

$$F_1(x) = \phi(x) + \psi(x).$$

La série (6) est convergente quel que soit x, mais la série (7) n'est convergente que si le module de x est plus grand que l'unité. Il résulte de là qu'il n'existe pas de fonction entière de z satisfaisant aux relations (4). Les deux fonctions $\phi(x)$ et $\psi(x)$ vérifient, respectivement, les deux relations

$$(tx-1)\,\phi(tx) = -\,2 + \phi(x), \tag{8}$$

$$(tx-1)\,\psi(tx) = \quad 2 + \psi(x). \tag{9}$$

La fonction $\psi(x)$ n'est définie, d'après ce qui précède, que pour les valeurs de x de module plus grand que 1. Pour achever de définir cette fonction, considérons la série

$$\psi(x) = 2Q_1.\left[\frac{1}{x-1} + \sum_{n=1}^{n=\infty} \frac{(-1)^n.t^n}{(1-t)(1-t^2)\ldots(1-t^n)}\cdot\frac{1}{x-t^n}\right]. \tag{10}$$

Cette série est convergente pour toutes les valeurs de x, sauf pour $x=0$ et pour certaines valeurs particulières $x=1$, $x=t$, etc., qui rendent infini un des termes. Je vais montrer que cette série (10) est identique à la série (7) lorsque le module de x est plus grand que l'unité; et dans la suite lorsque je parlerai de la fonction $\psi(x)$, j'entendrai par là la fonction (10), me rappelant seulement que cette fonction $\psi(x)$ peut être représentée par la série (7) lorsque mod. $x > 1$.

Pour montrer l'identité des séries (7) et (10) lorsque mod. $x > 1$ je remarque que, dans ce cas, la série (10) peut être ordonnée suivant les puissances positives croissantes de $\frac{1}{x}$; et que les deux séries vérifient la première la relation (9), la deuxième une relation de la forme

$$(tx-1)\,\psi(tx) = 2h + \psi(x), \tag{9$^{\text{bis}}$}$$

h désignant une constante exprimée par une série en t: c'est ce qu'on voit sans peine sur le développement (10). En cherchant par la méthode des coefficients indéterminés la fonction la plus générale de la forme

$$\frac{a_1}{x} + \frac{a_2}{x^2} + \ldots\ldots + \frac{a_n}{x^n} + \ldots\ldots$$

vérifiant la relation (9^{bis}), on trouve que cette fonction est égale à la série (7) multipliée par la constante h. La fonction $\psi(x)$ définie par la relation (10) est donc égale à la fonction $\psi(x)$ définie par la relation (7) multipliée par h. Mais si on multiplie la série (7) par $(x-1)$ et qu'on fasse tendre x vers 1, le produit tend vers la limite $2\prod_{n=1}^{n=\infty}(1+t^n)$, c'est à dire $2Q_1$, d'après un théorême que j'ai démontré,* et il en est de même évidemment de la série (10); donc les deux séries sont identiques, et $h=1$.

En revenant à mon problême, j'ai une fonction méromorphe satisfaisant aux relations (6) en prenant

$$F(z)=\phi\left(e^{\frac{2\pi zi}{\omega}}\right)+\psi\left(e^{\frac{2\pi zi}{\omega}}\right) \tag{11}$$

où

$$\phi\left(e^{\frac{2\pi zi}{\omega}}\right)=1+\sum_{n=1}^{n=\infty}\frac{q^{n(n+1)}}{(1+q^2)\ldots\ldots(1+q^{2n})}e^{\frac{2n\pi zi}{\omega}},$$

$$\psi\left(e^{\frac{2\pi zi}{\omega}}\right)=2Q_1\left[\frac{1}{e^{\frac{2\pi zi}{\omega}}-1}+\sum_{n=1}^{n=\infty}\frac{(-1)^n q^{2n}}{(1-q^2)\ldots\ldots(1-q^{2n})}\cdot\frac{1}{e^{\frac{2\pi zi}{\omega}}-q^{2n}}\right].$$

Si alors on pose

$$\Phi(z)=\frac{F(z)}{G(z)},$$

$F(z)$ et $G(z)$ étant les fonctions (11) et (3), cette fonction $\Phi(z)$ est doublement périodique, car elle vérifie les relations

$$\Phi(z+\omega)=\Phi(z),\quad \Phi(z+\omega')=-\Phi(z).$$

Cherchons les infinis de cette fonction. Le dénominateur $G(z)$ étant égal à $\dfrac{1}{\Omega\left(q^2,\frac{z}{\omega'}\right)}$ d'après les notations de Heine a pour zéros les valeurs

$$l\omega'+p\omega\begin{pmatrix}l=-1,\ldots,\ l=-\infty\\ p=-\infty,\ldots,p=+\infty\end{pmatrix};$$

le numérateur $F(z)$ se compose d'une partie entière $\phi\left(e^{\frac{2\pi zi}{\omega}}\right)$ et d'une partie méromorphe qui a pour infinis les valeurs

$$l\omega'+p\omega\begin{pmatrix}l=0\quad,\ldots,\ l=+\infty\\ p=-\infty,\ldots,p=+\infty\end{pmatrix}.$$

---

* Comptes Rendus, tome LXXXVII, page 689.

Donc les infinis de $\Phi(z)$ sont $(l\omega' + p\omega)$, l et p étant deux entiers qui reçoivent toutes les valeurs possibles. Or la fonction $\Theta_1(z)$ admet les mêmes valeurs pour zéros; donc le produit

$$\Psi(z) = \Theta_1(z)\,\Phi(z)$$

est une fonction *entière* qui satisfait aux relations

$$\Psi(z+\omega) = -\Psi(z),$$
$$\Psi(z+\omega') = \frac{1}{q}\,e^{-\frac{2\pi z i}{\omega}}\,\Psi(z)$$

par conséquent, en employant les notations de l'ouvrage de Briot et Bouquet

$$\Psi(z) = C\Theta_2(z),$$

C étant une constante. La fonction doublement périodique $\Phi(z)$ est, de cette façon, exprimée à l'aide des fonctions Θ, et l'on a

$$\frac{F(z)}{G(z)} = C\,\frac{\Theta_2(z)}{\Theta_1(z)}. \tag{12}$$

Pour déterminer C, multiplions les deux membres de la relation (12) par z, puis faisons tendre z vers 0. Le produit $zF(z)$ tend vers $\frac{\omega}{\pi i}\,Q_1$, et comme $G(0)$ est égal à 1, et $\frac{\Theta_1'(0)}{\Theta_2(0)}$ à $\sqrt{k'}$, on a:

$$C = \sqrt{k'}\,\frac{\omega}{\pi i}\,Q_1.$$

On a donc enfin pour la fonction elliptique $\nu(z)$ la formule

$$\frac{F(z)}{G(z)} = \frac{\omega}{\pi}\,Q_1 \cdot \nu\left(z + \frac{\omega'}{2}\right).$$

L'on obtient ainsi des expressions nouvelles des trois fonctions elliptiques $\lambda(z)$, $\mu(z)$, $\nu(z)$ au moyen du quotient de deux fonctions définies par des séries convergentes pour toutes les valeurs de z: il suffit, pour avoir $\lambda(z)$ et $\mu(z)$ de transformer la formule ci-dessus en ajoutant ou retranchant à z des demi-périodes.

Fourth Memoir on a New Theory of Symmetric Functions.

BY MAJOR P. A. MACMAHON, R. A., F. R. S.

---

§15.

194. At the conclusion of the previous memoir I applied the linear operations

$$g_0,\ g_1,\ g_{-1},\ \dots .$$

directly to the theory of separations; I proceed to the further development of this part of the subject; it will be merely necessary in general to consider symmetric functions symbolized by positive non-zero integers.

A previous result, given in §14, may be written

$$\frac{(-)^s}{s} g_s = \sum_\pi \sum_p \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\dots .\ \pi_3!\ \pi_2!\ \pi_1!} (\dots .\ 3^{p_3}2^{p_2}1^{p_1})\, \partial_{(\dots .\ 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1})};$$

the right-hand side may be broken up into fragments, in each of which the numbers

$$\dots .\ \pi_3,\ \pi_2,\ \pi_1$$

are constant.

We may thus write

$$\frac{(-)^s g_s}{s} = \sum_\pi \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\dots .\ \pi_3!\ \pi_2!\ \pi_1!} \sum_p (\dots .\ 3^{p_3}2^{p_2}1^{p_1})\, \partial_{(\dots .\ 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1})},$$

where the linear operator

$$\sum_p (\dots .\ 3^{p_3}2^{p_2}1^{p_1})\, \partial_{(\dots .\ 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1})},$$

in which the numbers

$$\dots .\ \pi_3,\ \pi_2,\ \pi_1$$

are constant, and the summation is merely in regard to the numbers

$$\dots .\ p_3,\ p_2,\ p_1,$$

that is, to every separate

$$(\ldots\ 3^{p_3+\pi_3}\,2^{p_2+\pi_2}\,1^{p_1+\pi_1}),$$

is one of the fragments above mentioned.

This fragmentary operation has of course a weight s; but further it may be regarded as having a partition

$$(\ldots\ 3^{\pi_3}2^{\pi_2}1^{\pi_1})$$

of the number s; we may so define the fragment and may write, for brevity and convenience,

$$\sum_p (\ldots\ 3^{p_3}2^{p_2}1^{p_1})\,\partial_{(\ldots 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1})} = g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1}} .$$

The operator relation is now written

$$\frac{(-)^s g_s}{s} = \sum_\pi \frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ \pi_3!\ \pi_2!\ \pi_1!}\,g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},$$

the summation being in regard to every partition

$$(\ldots\ 3^{\pi_3}2^{\pi_2}1^{\pi_1}),$$

of the number s, which occurs in the given separable partition.

195. Considering a perfectly general separable partition, every partition of s may occur, and it is, in consequence, convenient to discuss the full result

$$\frac{(-)^s}{s} g_s = \sum \frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ \pi_3!\ \pi_2!\ \pi_1!}\,g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},$$

the summation having reference to every partition of the number s.

196. Just as in the ordinary theory we meet with a linear differential operation corresponding to every number, so here in the wider theory of separations we are brought face to face with a linear differential operation which is in correspondence with an arbitrary partition of an arbitrary number.

197. In the simplest cases we have the equivalences

$$g_1 = g_{(1)},$$
$$g_2 = g_{(1^2)} - 2g_{(2)},$$
$$g_3 = g_{(1^3)} - 3g_{(21)} + 3g_{(3)},$$
$$g_4 = g_{(1^4)} - 4g_{(21^2)} + 2g_{(2^2)} + 4g_{(31)} - 4g_{(4)},$$

. . . .

each *weight operator* $g_1, g_2, g_3, g_4, \ldots$ being expressible as a linear function of its fragmentary *partition operators* according to the same law as the sums of powers are represented by elementary (that is, unitary) symmetric functions.

198. Let $$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},\quad g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})},$$

be any two partition operators of the same or of different weights. We have the known theorem:

$$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\, g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \overline{g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\, g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}} + g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} \dagger g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})},$$

where the bar written over the product on the right denotes that the operators are to be multiplied together symbolically and the symbol $\dagger$ denotes the performance of the operation $g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}$ upon the operator $g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}$, where the latter is considered to be a function of symbols of quantity only.

Now since

$$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} = \sum (\ldots\ldots 3^{j_3}2^{j_2}1^{j_1})\, \partial_{(\ldots 3^{j_3+\pi_3}2^{j_2+\pi_2}1^{j_1+\pi_1})},$$

$$g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \sum (\ldots\ldots 3^{j_3+\pi_3}2^{j_2+\pi_2}1^{j_1+\pi_1})\, \partial_{(\ldots 3^{j_3+\pi_3+\rho_3}2^{j_2+\pi_2+\rho_2}1^{j_1+\pi_1+\rho_1})},$$

there results

$$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} \dagger g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = g_{(\ldots 3^{\pi_3+\rho_3}2^{\pi_2+\rho_2}1^{\pi_1+\rho_1})},$$

and, thence, the multiplication theorem

$$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\, g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \overline{g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\, g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}} + g_{(\ldots 3^{\pi_3+\rho_3}2^{\pi_2+\rho_2}1^{\pi_1+\rho_1})},$$

analogous to the known theorem

$$g_s g_t = \overline{g_s g_t} + g_{s+t},$$

with which it should be contrasted.

199. We are now led to the result:

$$g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\, g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} - g_{(\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}\, g_{(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} = 0.$$

The expression on the left has been termed by Sylvester (in his Lectures on Reciprocants and elsewhere) the alternant of the operators involved.

Theorem. The *alternant* of any two *partition operators* vanishes.

200. This theorem again leads us to two corollaries—

Corollary 1. The *alternant* of any *partition operator* and any weight operator vanishes.

Corollary 2. The *alternant* of any two weight operators vanishes.

This last corollary is already well known.

The theorem may be otherwise stated by enunciating a necessary consequence.

201. *Theorem.* Any partition operator and any weight operator is commutable with any other partition or weight operator.

202. Consider now the solutions of the linear partial differential equation

$$P=0,$$

where P is any partition or weight operator.

If ϕ be one solution, so that identically

$$P\phi=0,$$

it follows that $Q\phi$ must be another solution, where Q is any other partition or weight operator.

For $$PQ\phi-QP\phi=0,$$

and since $$P\phi=0,$$

therefore also $$P(Q\phi)=0,$$

or $Q\phi$ is also a solution of

$$P=0.$$

203. *Theorem.* If ϕ be a solution of $P=0$, $Q\phi$ is also a solution when P and Q are any two partition or weight operations.

204. The partition operators are of most importance in the case of the differential equation

$$g_s=0.$$

For we have seen that

$$\frac{(-)^s g_s}{s}=\sum_{\pi}\frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ldots\pi_3!\,\pi_2!\,\pi_1!}g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},$$

and if $g_s\phi=0$, where ϕ is expressed in terms of separations of the separable partition

$$(\ldots\ldots 3^{P_3}2^{P_2}1^{P_1}),$$

the effect of the partition operator

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}$$

will be the production of terms each of which is a separation of the partition

$$(\ldots\, 3^{p_3-\pi_3}2^{p_2-\pi_2}1^{p_1-\pi_1}).$$

No other partition operator can produce separations of this partition. Since therefore

$$g_s\phi$$

is identically zero, we must have also

$$g_{(\ldots\, 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,\phi = 0\,.$$

205. *Theorem.* If a function be annihilated by a *weight operator*, it must also be annihilated by each *partition operator* of that weight.

This important and comprehensive theorem renders the calculation of tables of separations a straightforward and comparatively easy matter.

206. As an example, suppose we have to calculate the function (3^2) in terms of separations of (31^3).

Assume

$$2(3^2) = A(31^3) + B(31^2)(1) + C(3)(1^3) + D(31)(1^2) + E(31)(1)^3 + F(3)(1^2)(1),$$

there being of necessity no term $(3)(1)^3$.

We have

$$\begin{aligned}
g_{(1)} &= \partial_{(1)} + (1)\,\partial_{(1^2)} + (1^2)\,\partial_{(1^3)} + (3)\,\partial_{(31)} + (31)\,\partial_{(31^2)} + (31^2)\,\partial_{(31^3)},\\
g_{(1^2)} &= \partial_{(1^2)} + (1)\,\partial_{(1^3)} + (3)\,\partial_{(31^2)} + (31)\,\partial_{(31^3)},\\
g_{(1^3)} &= \partial_{(1^3)} + (3)\,\partial_{(31^3)},\\
g_{(3)} &= \partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (1^3)\,\partial_{(31^3)},\\
g_{(31)} &= \partial_{(31)} + (1)\,\partial_{(31^2)} + (1^2)\,\partial_{(31^3)},\\
g_{(31^2)} &= \partial_{(31^2)} + (1)\,\partial_{(31^3)},\\
g_{(31^3)} &= \partial_{(31^3)},
\end{aligned}$$

and these are the only operations we are concerned with; their number is $(1+1)(3+1)-1=7$, viz. one for each separate. The weight operators which annihilate the function are g_1, g_2, g_4 and g_5. Hence we have as annihilators the partition operators

$$g_{(1)},\; g_{(1^2)},\; g_{(31)},\; g_{(31^2)}\,.$$

Further,

$$\begin{aligned}
g_3 2(3^2) &= -\,6\,(3),\\
g_6 2(3^2) &= +\,6\,,
\end{aligned}$$

that is,

$$[\partial_{(1^3)} + (3)\,\partial_{(31^2)} + 3\{\partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (1^3)\,\partial_{(31^3)}\}]\,2\,(3^2) = -\,6\,(3),$$
$$6\,\partial_{(31^3)}\,2\,(3^2) = +\,6\,.$$

These operations are more than sufficient to determine all the coefficients and to verify the result to be found in Vol. XI, p. 17 of this Journal.

§16.

207. Recalling the equivalences

$$\begin{aligned}
g_1 &= g_{(1)}, \\
g_2 &= g_{(1^2)} - 2g_{(2)}, \\
g_3 &= g_{(1^3)} - 3g_{(21)} + 3g_{(3)}, \\
g_4 &= g_{(1^4)} - 4g_{(21^2)} + 2g_{(2^2)} + 4g_{(31)} - 4g_{(4)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

I find it convenient to write them in a new notation, so as to bring into better evidence their law of formation. I write

$$\begin{aligned}
g_{[1]} &= g_{[1]}, \\
g_{[2]} &= g_{[1]^2} - 2g_{[1^2]}, \\
g_{[3]} &= g_{[1]^3} - 3g_{[1^2][1]} + 3g_{[1^3]}, \\
g_{[4]} &= g_{[1]^4} - 4g_{[1^2][1]^2} + 2g_{[1^2]^2} + 4g_{[1^3][1]} - 4g_{[1^4]}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

where $g_{[1^\lambda][1^\mu]\ldots\ldots}$ takes the place of $g_{(\lambda\mu\ldots\ldots)}$ and $g_{[\lambda]}$ that of g_λ. This notation is quite consistent. In general $g_{[\lambda\ldots\ldots][\mu\ldots\ldots]\ldots\ldots}$ denotes that the operator is formed according to the same law as the symmetric function $(\lambda\ldots\ldots)(\mu\ldots\ldots)\ldots\ldots$ when expressed in terms of elementary symmetric functions.

The theorems

$$\begin{aligned}
g_\lambda \dagger g_\mu &= g_{\lambda+\mu}, \\
g_{(\lambda\mu\ldots\ldots)} \dagger g_{(\pi\rho\ldots\ldots)} &= g_{(\lambda\mu\ldots\ldots\pi\rho\ldots\ldots)}
\end{aligned}$$

now become in the new notation

$$\begin{aligned}
g_{[\lambda]} \dagger g_{[\mu]} &= g_{[\lambda][\mu]}, \\
g_{[1^\lambda][1^\mu]\ldots\ldots} \dagger g_{[1^\pi][1^\rho]\ldots\ldots} &= g_{[1^\lambda][1^\mu]\ldots\ldots[1^\pi][1^\rho]\ldots\ldots},
\end{aligned}$$

that is, a partition suffix addition is transformed into a partition suffix multiplication.

208. We have now in succession

$$g_{[\lambda]}g_{[\mu]} = \overline{g_{[\lambda]}g_{[\mu]}} + g_{[\lambda][\mu]},$$
$$g_{[\kappa]}g_{[\lambda]}g_{[\mu]} = \overline{g_{[\kappa]}g_{[\lambda]}g_{[\mu]}} + \overline{g_{[\kappa]}g_{[\lambda][\mu]}} + \overline{g_{[\lambda]}g_{[\mu][\kappa]}} + \overline{g_{[\mu]}g_{[\kappa][\lambda]}} + g_{[\kappa][\lambda][\mu]},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

which are collateral with the symmetric function theorems:

$$(\lambda)(\mu) = (\lambda\mu) + (\lambda + \mu),$$
$$(\kappa)(\lambda)(\mu) = (\kappa\lambda\mu) + (\kappa, \lambda + \mu) + (\lambda, \mu + \kappa) + (\mu, \kappa + \lambda) + (\kappa + \lambda + \mu),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

and further,

$$\overline{g_{[\lambda]}g_{[\mu]}} = g_{[\lambda]}g_{[\mu]} - g_{[\lambda][\mu]},$$
$$\overline{g_{[\kappa]}g_{[\lambda]}g_{[\mu]}} = g_{[\kappa]}g_{[\lambda]}g_{[\mu]} - g_{[\kappa]}g_{[\lambda][\mu]} - g_{[\lambda]}g_{[\mu][\kappa]} - g_{[\mu]}g_{[\kappa][\lambda]} + 2g_{[\kappa][\lambda][\mu]},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

in correspondence with the known results:

$$(\lambda\mu) = s_\lambda s_\mu - s_{\lambda+\mu},$$
$$(\kappa\lambda\mu) = s_\kappa s_\lambda s_\mu - s_\kappa s_{\lambda+\mu} - s_\lambda s_{\mu+\kappa} - s_\mu s_{\kappa+\lambda} + 2s_{\kappa+\lambda+\mu},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

209. In both systems of formulae the same modifications, in particular cases, are necessary.

N. B.—Note the result,

$$g_{[\lambda][\mu]} = g_{[\lambda\mu]} + g_{[\lambda+\mu]},$$

and so on; similar developments proceed exactly as in the case of symmetric functions.

§17.

210. Reverting again to the previous notation,

$$g_1 = g_{(1)},$$
$$g_2 = g_{(1^2)} - 2g_{(2)},$$
$$\text{etc.} = \text{etc.},$$

and recalling the well-known relations, viz.

$$g_1 = G_1,$$
$$g_2 = G_1^2 - 2G_2,$$
$$g_3 = G_1^3 - 3G_2G_1 + 3G_3,$$
$$\text{etc.} = \text{etc.},$$

p14725.

where $G_1, G_2, G_3 \ldots$ are the obliterators before met with, we notice the similarity of the two laws of operation.

211. We deduce

$$\begin{aligned}
G_1 &= g_1 = g_{(1)}, \\
2G_2 &= g_1^2 - g_2 = g_{(1)}^2 - g_{(1^2)} + 2g_{(2)}, \\
6G_3 &= g_1^3 - 3g_2g_1 + 2g_3, \\
&= g_{(1)}^3 - 3g_{(1^2)}g_{(1)} + 2g_{(1^3)} + 6\{g_{(2)}g_{(1)} - g_{(21)}\} + 6g_{(3)}, \\
24G_4 &= g_1^4 - 6g_2g_1^2 + 3g_2^2 + 8g_3g_1 - 6g_4, \\
&= g_{(1)}^4 - 6g_{(1^2)}g_{(1)}^2 + 3g_{(1^2)}^2 + 8g_{(1^3)}g_{(1)} - 6g_{(1^4)} \\
&\quad + 12\{g_{(2)}g_{(1)}^2 - g_{(2)}g_{(1^2)} - 2g_{(21)}g_{(1)} + 2g_{(21^2)}\} + 12\{g_{(2)}^2 - g_{(2^2)}\} \\
&\quad + 24\{g_{(3)}g_{(1)} - g_{(31)}\} + 24g_{(4)},
\end{aligned}$$

after some slight reduction.

212. We can thus always express the obliterators $G_1, G_2, G_3, \ldots$ in terms of successive operations of the linear partition operators.

To see the law, on the right-hand side of the identity, last obtained, replace each partition by a partition containing a single part equal in magnitude to the weight of the partition and omit the literal symbols altogether. We thus obtain

$$\begin{aligned}
\{(1)^4 - 6(2)(1)^2 + 3(2)^2 + 8(3)(1) - 6(4)\} + 12\{(2)(1)^2 - (2)^2 - 2(3)(1) + 2(4)\} \\
+ 12\{(2)^2 - (4)\} + 24\{(3)(1) - (4)\} + 24(4),
\end{aligned}$$

which is

$$\begin{aligned}
24(1^4) + 12.2(21^2) + 12.2(2^2) + 24(31) + 24(4) \\
= 24\{(1^4) + (21^2) + (2^2) + (31) + (4)\}.
\end{aligned}$$

It is easy to establish *à priori* that this must be so. The expression of $24G_4$ breaks up into 5 portions corresponding to the 5 partitions of the number 4. In general the expression for G_s breaks up into as many portions as there are partitions of s, and in general

$$G_s = \sum \frac{(-)^{L_1+L_2+\ldots+l+m+\ldots}}{L_1!\,L_2!\ldots} \left\{\frac{(l_1+m_1+\ldots-1)!}{l_1!\,m_1!\ldots}\right\}^{L_1} \left\{\frac{(l_2+m_2+\ldots-1)!}{l_2!\,m_2!\ldots}\right\}^{L_2} \ldots g^{L_1}_{(\lambda_1^{l_1}\mu_1^{m_1}\ldots)} g^{L_2}_{(\lambda_2^{l_2}\mu_2^{m_2}\ldots)} \ldots,$$

where $(\lambda_1^{l_1}\mu_1^{m_1}\ldots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots)^{L_2}\ldots$

is any separation of any partition

$$(\lambda^l\mu^m\ldots)$$

of the number s.

213. Instead of expressing G_s in terms of successive operations of linear partition operators, we may express it in terms of operators formed by multiplying together the partition operators symbolically. The system of relations is

$$\begin{aligned} G_1 &= g_{(1)}, \\ 2G_2 &= \overline{g^2_{(1)}} + 2g_{(2)}, \\ 6G_3 &= \overline{g^3_{(1)}} + 6\overline{g_{(2)}g_{(1)}} + 6g_{(3)}, \\ 24G_4 &= \overline{g^4_{(1)}} + 12\overline{g_{(2)}g^2_{(1)}} + 12\overline{g^2_{(2)}} + 24\overline{g_{(3)}g_{(1)}} + 24g_{(4)}, \\ \text{etc.} &= \text{etc.}, \end{aligned}$$

and in general

$$G_s = \sum \frac{1}{l!\, m! \ldots .} \overline{g^l_{(\lambda)} g^m_{(\mu)} \ldots .}$$

§18.

214. In what has preceded in respect of the linear operations $g_1, g_2, g_3, \ldots .$ a generalization has been made from a number to the partition of a number and weight operators were broken up into linear functions of partition operators.

A like generalization can be made in respect of the obliterating operators $G_1, G_2, G_3, \ldots .$

215. Suppose a symmetric function

$$f(a_1, a_2, a_3, \ldots . a_s, \ldots .) = f$$

to be the product of m monomial functions, and write

$$f = f_1 f_2 f_3 \ldots . f_m.$$

If a_s be changed into $a_s + \mu a_{s-1}$, we have from previous work

$$\begin{aligned} (1 + \mu G_1 + \mu^2 G_2 + \mu^3 G_3 + \ldots . + \mu^s G_s + \ldots .) f & \\ = (1 + \mu G_1 + \mu^2 G_2 + \ldots . + \mu^s G_s + \ldots .) f_1 & \\ \times (1 + \mu G_1 + \mu^2 G_2 + \ldots . + \mu^s G_s + \ldots .) f_2 & \\ \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \times (1 + \mu G_1 + \mu^2 G_2 + \ldots . + \mu^s G_s + \ldots .) f_m, & \end{aligned}$$

and now expanding and equating coefficients of like powers of μ, there result:

$$\begin{aligned} G_1 f &= \Sigma (G_1 f_1) f_2 f_3 \ldots . f_m, \\ G_2 f &= \Sigma (G_1 f_1)(G_1 f_2) f_3 \ldots . f_m + \Sigma (G_2 f_1) f_2 \ldots . f_m, \\ G_3 f_1 &= \Sigma (G_1 f_1)(G_1 f_2)(G_1 f_3) f_4 \ldots . f_m \\ &\quad + \Sigma (G_2 f_1)(G_1 f_2) f_3 \ldots . f_m + \Sigma (G_3 f_1) f_2 f_3 \ldots . f_m, \end{aligned}$$

and so on, the summations being in regard to the different terms obtained by permutation of the m suffixes of the functions $f_1, f_2, \ldots . f_m$.

216. In general, in the expression of G_s there appears a summation corresponding to each partition of the number s.

The summation in correspondence with a partition $(p_1, p_2, \ldots . p_s)$ is

$$\Sigma (G_{p_1} f_1)(G_{p_2} f_2) \ldots . (G_{p_s} f_s) f_{s+1} \ldots . f_m .$$

Thus, when performed upon a product of functions, the operator G_s breaks up into as many distinct operations as the weight s possesses partitions.

I denote the operation indicated by the summation

$$\Sigma (G_{p_1} f_1)(G_{p_2} f_2) \ldots . (G_{p_s} f_s) f_{s+1} \ldots . f_m$$

by

$$G_{(p_1 p_2 \ldots . p_s)},$$

and speak of it as a partition obliterating operator.

217. We have now the equivalence

$$G_s = \Sigma G_{(p_1 p_2 \ldots . p_s)},$$

the summation being in regard to every partition of the weight s.

This theorem indicates the method of operating with G_s upon a product of symmetric functions.

In particular,

$$\begin{aligned} G_1 &= G_{(1)}, \\ G_2 &= G_{(1^2)} + G_{(2)}, \\ G_3 &= G_{(1^3)} + G_{(21)} + G_{(3)}, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

218. Interesting relations may be established between the partition g and the partition G operators.

From the relation

$$\sum_{\pi} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots .}\, g_{(p_1^{\pi_1} p_2^{\pi_2} \ldots .)} = \sum_{\pi} \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots .}\, G_{p_1}^{\pi_1} G_{p_2}^{\pi_2} \ldots .$$

there arise the relations

$$\begin{aligned} g_{(1)} &= G_1 = G_{(1)}, \\ g_{(1^2)} - 2g_{(2)} &= G_1^2 - 2G_2 = G_{(1)}^2 - 2G_{(1^2)} - 2G_{(2)}, \\ g_{(1^3)} - 3g_{(21)} + 3g_{(3)} &= G_1^3 - 3G_2G_1 + 3G_3 \\ &= G_{(1)}^3 - 3G_{(1^2)}G_{(1)} + 3G_{(1^3)} - 3\{G_{(2)}G_{(1)} - G_{(21)}\} + 3G_{(3)}, \end{aligned}$$

and so forth.

219. Considering particularly the relation last written, I say that it may be broken up into three relations, viz.

$$\begin{aligned} g_{(1^3)} &= G^3_{(1)} - 3G_{(1^2)}G_{(1)} + 3G_{(1^3)}, \\ g_{(21)} &= G_{(2)}G_{(1)} - G_{(21)}, \\ g_{(3)} &= G_{(3)}, \end{aligned}$$

for it is easy to see that the two sides of the unfractured operator relation must produce upon any operand the same result identically. Hence after operation those functions which are separations of the same function must separately vanish, and hence we must have the equivalences of operations above set forth.

220. In general there exists the relation

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\,\ldots.}\,g_{(p_1^{\pi_1}p_2^{\pi_2}\ldots.)} = \sum_j \frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\,\ldots.}\,G^{j_1}_{(J_1)}G^{j_2}_{(J_2)}\ldots.,$$

the summation being for all separations

$$(J_1)^{j_1}(J_2)^{j_2}\ldots.$$

of the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$.

221. This result gives the general relation between the partition g and the partition G operators, and should be compared with the formula which expresses a function symbolized by a single part in terms of separations of $(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$.

222. The relation when reversed is

$$\begin{aligned} &(-)^{\Sigma\pi-1}G_{(p_1^{\pi_1}p_2^{\pi_2}\ldots.)} \\ &\quad = \sum_j \frac{(-)^{\Sigma j-1}(\Sigma\pi_1-1)!\,(\Sigma\pi_2-1)!\,\ldots.}{j_1!\,j_2!\,\ldots.\,\pi_{11}!\,\pi_{12}!\,\ldots.\,\pi_{21}!\,\pi_{22}!\,\ldots.}\,g^{j_1}_{(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots.)}g^{j_2}_{(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots.)}\ldots., \end{aligned}$$

the summation being for every separation of the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots.)$.

223. The following results of operations should be remarked:

$$G_{(p_1^{\pi_1}p_2^{\pi_2}\ldots.)}s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\ldots. = 1,$$

$$\frac{1}{j_1!}\,\frac{1}{j_2!}\ldots.\,g^{j_1}_{(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots.)}g^{j_2}_{(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots.)}\ldots.\,(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots.)^{j_1}(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots.)^{j_2}\ldots. = 1.$$

§19.

224. There is a more extensive law of reciprocity than that established in §10 of the Third Memoir. The latter sprang from three identities of the form

$$\begin{aligned} &1+A_0+A_1x+A_2x^2+\ldots = e^n(1+\alpha_1x)(1+\alpha_2x)\ldots\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\ldots, \\ &\quad +A_{-1}\frac{1}{x}+A_{-2}\frac{1}{x^2}+\ldots \end{aligned}$$

which for brevity may be written

$$F(A) = f(\alpha).$$

225. We may instead consider any number of such identities; but first of all, for the sake of simplicity, let us consider four identities

$$\begin{aligned} F(A) &= f(\alpha), \\ F(B) &= f(\beta), \\ F(C) &= f(\gamma), \\ F(D) &= f(\delta), \end{aligned}$$

and add the auxiliary identity

$$F(K) = f(\varkappa).$$

226. Assume the quantities herein involved to be connected by the two relations

$$\begin{aligned} &1 + \mathrm{K}_0 + \mathrm{K}_1 y + \mathrm{K}_2 y^2 + \ldots \\ &\quad + \mathrm{K}_{-1}\frac{1}{y} + \mathrm{K}_{-2}\frac{1}{y^2} + \ldots = \Pi_s \left(\begin{aligned} &1 + B_0 + \alpha_s B_1 y + \alpha_s^2 B_2 y^2 + \ldots \\ &\quad + \frac{1}{\alpha_s} B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2} B_{-2}\frac{1}{y^2} + \ldots \end{aligned} \right); \\ &1 + D_0 + D_1 y + D_2 y^2 + \ldots \\ &\quad + D_{-1}\frac{1}{y} + D_{-2}\frac{1}{y^2} + \ldots = \Pi_s \left(\begin{aligned} &1 + \mathrm{K}_0 + \gamma_s \mathrm{K}_1 y + \gamma_s^2 \mathrm{K}_2 y^2 + \ldots \\ &\quad + \frac{1}{\gamma_s}\mathrm{K}_{-1}\frac{1}{y} + \frac{1}{\gamma_s^2}\mathrm{K}_{-2}\frac{1}{y^2} + \ldots \end{aligned} \right). \end{aligned}$$

Denoting $\Sigma\alpha_1^m$ by $(m)_\alpha$, it has been shown in §10 that these relations lead to the identities

$$\begin{aligned} (m)_\varkappa &= (m)_\alpha (m)_\beta, \\ (m)_\delta &= (m)_\varkappa (m)_\gamma, \end{aligned}$$

so that eliminating $(m)_\varkappa$ we have

$$(m)_\delta = (m)_\alpha (m)_\beta (m)_\gamma;$$

this is equivalent to the result of the elimination of the quantities K between the two assumed relations; in fact we find immediately

$$\begin{aligned} &1 + D_0 + D_1 y + D_2 y^2 + \ldots \\ &\quad + D_{-1}\frac{1}{y} + D_{-2}\frac{1}{y^2} + \ldots = \Pi_s \Pi_t \left(\begin{aligned} &1 + B_0 + \alpha_s\gamma_t B_1 y + \alpha_s^2\gamma_t^2 B_2 y^2 + \ldots \\ &\quad + \frac{1}{\alpha_s\gamma_t} B_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2\gamma_t^2} B_{-2}\frac{1}{y^2} + \ldots \end{aligned} \right), \end{aligned}$$

and the result is then reached by taking logarithms and expanding in powers of y.

In the found relation

$$(m)_\delta = (m)_\alpha (m)_\beta (m)_\gamma,$$

m may be any integer, positive, zero or negative.

227. The expression $(m)_\delta$ remains unchanged for any permutation of the sets of quantities

$$\begin{array}{l} \alpha_1, \alpha_2, \alpha_3, \dots . \\ \beta_1, \beta_2, \beta_3, \dots . \\ \gamma_1, \gamma_2, \gamma_3, \dots . \end{array}$$

and hence every symmetric function of the quantities

$$\delta_1, \delta_2, \delta_3, \dots .$$

remains unchanged for all permutations amongst the three sets mentioned.

228. Thus if we have found

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \dots .)_\delta = \dots . + J (p_1^{\pi_1} p_2^{\pi_2} \dots .)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \dots .)_\beta (\mu_1^{m_1} \mu_2^{m_2} \dots .)_\gamma + \dots . ,$$

we necessarily have

$$(s_1^{\sigma_1} s_2^{\sigma_2} \dots .)_\delta = \dots . + J \{ (p_1^{\pi_1} p_2^{\pi_2} \dots .)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \dots .)_\beta (\mu_2^{m_1} \mu_1^{m_2} \dots .)_\gamma + 5 \text{ similar expressions obtained by permuting } \alpha, \beta \text{ and } \gamma \} + \dots .$$

229. The two assumed relations lead, as shown in §10, to the operator relations

$$\begin{aligned} {}_\beta g_m &= (m)_\alpha \, {}_\kappa g_m, \\ {}_\kappa g_m &= (m)_\gamma \, {}_\delta g_m, \end{aligned}$$

and thence

$$(m)_\delta \, {}_\delta g_m = (m)_\alpha \, {}_\alpha g_m = (m)_\beta \, {}_\beta g_m = (m)_\gamma \, {}_\gamma g_m,$$

showing the invariant character of the operation

$$(m)_\delta \, {}_\delta g_m,$$

for any transformation of a function of the quantities D into a function of either of the sets of quantities A, B, C as given by the relations assumed.

230. Since $${}_\beta g_m = (m)_\alpha (m)_\gamma \, {}_\delta g_m,$$
we may write

$$\begin{aligned} & {}_\beta g_0 + {}_\beta g_1 y - \tfrac{1}{2} \, {}_\beta g_2 y^2 + \dots . \\ & \quad + {}_\beta g_{-1} \frac{1}{y} - \tfrac{1}{2} \, {}_\beta g_{-2} \frac{1}{y^2} + \dots . \\ & \qquad = (0)_\alpha (0)_\gamma \, {}_\delta g_0 + (1)_\alpha (1)_\gamma \, {}_\delta g_1 y - \tfrac{1}{2} (2)_\alpha (2)_\gamma \, {}_\delta g_2 y^2 + \dots . \\ & \qquad\qquad + (\overline{1})_\alpha (\overline{1})_\gamma \, {}_\delta g_{-1} \frac{1}{y} - \tfrac{1}{2} (\overline{2})_\alpha (\overline{2})_\gamma \, {}_\delta g_{-2} \frac{1}{y^2} + \dots . , \end{aligned}$$

and by taking the exponential of each side, we reach by previous work the relation

$$\begin{aligned}1 + {}_\beta G_0 + {}_\beta G_1 y \quad + {}_\beta G_2 y^2 \quad + \ldots\ldots \\ + {}_\beta G_{-1}\frac{1}{y} + {}_\beta G_{-2}\frac{1}{y^2} + \ldots\ldots \\ = \prod_s \prod_t \Big(1 + {}_\delta G_0 + \alpha_s\gamma_t\,{}_\delta G_1 y \quad + \alpha_s^2\gamma_t^2\,{}_\delta G_2 y^2 \quad + \ldots\ldots \\ + \frac{1}{\alpha_s\gamma_t}\,{}_\delta G_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2\gamma_t^2}\,{}_\delta G_{-2}\frac{1}{y^2} + \ldots\ldots\Big),\end{aligned}$$

showing that in any relation, connecting the quantities D with the quantities B, we obtain an operator relation by writing

$$\begin{aligned} &{}_\beta G_m \text{ for } D_m, \\ \text{and} \qquad &{}_\delta G_m \text{ for } B_m. \end{aligned}$$

231. In the relation just established we may make any permutation of the letters α, β, γ, so that, regarding the quantities D as expressed in terms of the quantities A or C, we may write

$$\begin{aligned} &{}_\alpha G_m \text{ for } D_m, \\ &{}_\delta G_m \text{ for } A_m, \\ \text{or} \qquad &{}_\gamma G_m \text{ for } D_m, \\ &{}_\delta G_m \text{ for } C_m \end{aligned}$$

in either case.

232. Consider now the quantities D expressed in terms of the quantities C, so that by multiplication we obtain a relation such as

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2}\ldots\ldots = \ldots\ldots + L(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta C_{s_1}^{\sigma_1} C_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots \qquad \text{I}$$

and also two others such as

$$D_{\lambda_1}^{l_1} D_{\lambda_2}^{l_2}\ldots\ldots = \ldots\ldots + M(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta C_{s_1}^{\sigma_1} C_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots \qquad \text{II}$$

$$D_{\mu_1}^{m_1} D_{\mu_2}^{m_2}\ldots\ldots = \ldots\ldots + N(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_\beta C_{s_1}^{\sigma_1} C_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots \qquad \text{III}$$

233. Assume, moreover,

$$(s_1^{\sigma_1}s_2^{\sigma_2}\ldots\ldots)_\delta = \ldots\ldots + J\{(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)_\gamma + \ldots\ldots\} + \ldots\ldots \qquad \text{IV}$$

The relation I yields the operator relation

$${}_\gamma G_{p_1}^{\pi_1}\,{}_\gamma G_{p_2}^{\pi_2}\ldots\ldots = \ldots\ldots + L(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta\,{}_\delta G_{s_1}^{\sigma_1}\,{}_\delta G_{s_2}^{\sigma_2}\ldots\ldots + \ldots\ldots,$$

which, performed upon opposite sides of IV, gives

$$\ldots\ldots + L(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta = \ldots\ldots + J(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\ldots\ldots)_\beta + \ldots\ldots,$$

whence $L = J$,
and similarly, $M = N = J$;
that is, $L = M = N$.

234. Hence in the relation

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} \ldots \ldots = \ldots \ldots + L (\lambda_1^{l_1} \lambda_2^{l_2} \ldots \ldots)_\alpha (\mu_1^{m_1} \mu_2^{m_2} \ldots \ldots)_\beta C_{s_1}^{\sigma_1} C_{s_2}^{\sigma_2} \ldots \ldots + \ldots \ldots,$$

if any permutation be impressed upon the three partitions

$$(\lambda_1^{l_1} \lambda_2^{l_2} \ldots \ldots),\ (\mu_1^{m_1} \mu_2^{m_2} \ldots \ldots),\ (p_1^{\pi_1} p_2^{\pi_2} \ldots \ldots),$$

the numerical coefficient L remains unchanged.

235. More generally consider n identities

$$\begin{aligned} F(A_1) &= f(\alpha_1), \\ F(A_2) &= f(\alpha_2), \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ F(A_n) &= f(\alpha_n), \end{aligned}$$

and therewith, $n - 3$ auxiliary identities

$$\begin{aligned} F(\mathrm{K}_1) &= f(x_1), \\ F(\mathrm{K}_2) &= f(x_2), \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ F(\mathrm{K}_{n-3}) &= f(x_{n-3}). \end{aligned}$$

236. Assume to exist between the quantities involved $n - 2$ relations, viz.

$$\begin{aligned} 1 + \mathrm{K}_{1,0} + \mathrm{K}_{1,1} y + \mathrm{K}_{1,2} y^2 + \ldots & \\ + \mathrm{K}_{1,-1} \frac{1}{y} + \mathrm{K}_{1,-2} \frac{1}{y^2} + \ldots & = \Pi_s \begin{pmatrix} 1 + A_{1,0} + \alpha_{2,s} A_{1,1} y + \alpha_{2,s}^2 A_{1,2} y^2 + \ldots \\ + \frac{1}{\alpha_{2,s}} A_{1,-1} \frac{1}{y} + \frac{1}{\alpha_{2,s}^2} A_{1,-2} \frac{1}{y^2} + \ldots \end{pmatrix}, \end{aligned}$$

which for brevity write

$$\Phi(\mathrm{K}_1) = \phi(\alpha_2, \alpha_1),$$

and also

$$\begin{aligned} \Phi(\mathrm{K}_2) &= \phi(\alpha_3, x_1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \Phi(\mathrm{K}_{n-3}) &= \phi(\alpha_{n-2}, x_{n-4}), \\ \Phi(A_n) &= \phi(\alpha_{n-1}, x_{n-3}). \end{aligned}$$

237. Observe that the relation

$$\Phi(\mathrm{K}_s) = \phi(\alpha_{s+1}, x_{s-1}),$$

is symmetrical with regard to the sets of quantities

$$\alpha_{s+1,1},\ \alpha_{s+1,2},\ \alpha_{s+1,3},\ \ldots\ldots \text{ and } \varkappa_{s-1,1},\ \varkappa_{s-1,2},\ \varkappa_{s-1,3},\ \ldots\ldots$$

238. We can now eliminate all the auxiliary quantities.

We have

$$\begin{aligned} (m)_{\kappa_1} &= (m)_{\alpha_2}(m)_{\alpha_1}, \\ (m)_{\kappa_2} &= (m)_{\alpha_3}(m)_{\kappa_1}, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots \\ (m)_{\kappa_{n-3}} &= (m)_{\alpha_{n-2}}(m)_{\kappa_{n-4}}, \\ (m)_{\alpha_n} &= (m)_{\alpha_{n-1}}(m)_{\kappa_{n-3}}, \end{aligned}$$

and thence

$$(m)_{\alpha_n} = (m)_{\alpha_1}(m)_{\alpha_2}\ldots\ldots(m)_{\alpha_{n-1}},$$

and thence, as before, we obtain the identity

$$\begin{aligned} &1 + A_{n,0} + A_{n,1}y \quad + A_{n,2}y^2 \quad + \ldots\ldots \\ &\qquad + A_{n,-1}\frac{1}{y} + A_{n,-2}\frac{1}{y^2} + \ldots\ldots \\ &= \prod_{s_2}\ldots\prod_{s_{n-1}}\left(\begin{aligned} &1+A_{1,0}+\alpha_{2,s_2}\alpha_{3,s_3}\ldots\alpha_{n-1,s_{n-1}}A_{1,1}y+\alpha^2_{2,s_2}\alpha^2_{3,s_3}\ldots\alpha^2_{n-1,s_{n-1}}A_{1,2}y^2+\ldots \\ &+\frac{1}{\alpha_{2,s_2}\alpha_{3,s_3}\ldots\alpha_{n-1,s_{n-1}}}A_{1,-1}\frac{1}{y}+\frac{1}{\alpha^2_{2,s_2}\alpha^2_{3,s_3}\ldots\alpha^2_{n-1,s_{n-1}}}A_{1,-2}\frac{1}{y^2}+\ldots \end{aligned}\right) \end{aligned}$$

239. This result is invariant for every substitution that may be impressed upon the sets of quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ldots\ \alpha_{n-1}.$$

240. Proceeding by the same method, as in the preceding particular case, it is easy to prove that if the identity above given lead to a result

$$\begin{aligned} &A^{l_{n,1}}_{n,\lambda_{n,1}}A^{l_{n,2}}_{n,\lambda_{n,2}}\ldots\ldots = \ldots\ldots \\ &\qquad + L\,(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots)_{\alpha_2}(\lambda_{31}^{l_{31}}\lambda_{32}^{l_{32}}\ldots\ldots)_{\alpha_3}\ldots(\lambda_{n-1,1}^{l_{n-1,1}}\lambda_{n-1,2}^{l_{n-1,2}}\ldots)_{\alpha_{n-1}}A^{l_{11}}_{1,\lambda_{11}}A^{l_{12}}_{1,\lambda_{12}}\ldots + \ldots, \end{aligned}$$

wherein L is a numerical coefficient, the number L remains unaltered when any substitution is impressed upon the $n-1$ partitions

$$(\lambda_{s_1}^{l_{s1}}\lambda_{s_2}^{l_{s2}}\ldots\ldots);$$
$$s = 2,\ 3,\ \ldots\ldots n.$$

241. This theorem expresses a very general law of symmetry. For its further examination I shall restrict myself to partitions which contain positive integers, zero excluded. This restriction is made with the sole idea of avoiding complexity in the expressions.

242. Consider these relations

$$1 + A_{s,1}x + A_{s,2}x^2 + \ldots\ldots = (1 + \alpha_{s,1}x)(1 + \alpha_{s,2}x)\ldots\ldots,$$
$$(s = 1,\ 2,\ \ldots\ldots n),$$

and

$$1 + \mathrm{K}_{s,1}x + \mathrm{K}_{s,2}x^2 + \ldots\ldots = (1 + \varkappa_{s,1}x)(1 + \varkappa_{s,2}x)\ldots\ldots,$$
$$\{s = 1,\ 2,\ \ldots\ldots (n-3)\}.$$

243. The relations, assumed to exist between the quantities involved, become

$$1+\mathrm{K}_{1,1}y \quad +\mathrm{K}_{1,2}y^2 \quad + \ldots = \prod_{s_2}(1+\alpha_{2,s_2}A_{1,1}y \qquad + \alpha^2_{2,s_2}A_{1,2}y^2 \qquad + \ldots),$$
$$1+\mathrm{K}_{2,1}y \quad +\mathrm{K}_{2,2}y^2 \quad + \ldots = \prod_{s_3}(1+\alpha_{3,s_3}\mathrm{K}_{1,1}y \qquad + \alpha^2_{3,s_3}\mathrm{K}_{1,2}y^2 \qquad + \ldots),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$1+\mathrm{K}_{n-3,1}y + \mathrm{K}_{n-3,2}y^2 + \ldots = \prod_{s_{n-2}}(1+\alpha_{n-2,s_{n-2}}\mathrm{K}_{n-4,1}y + \alpha^2_{n-2,s_{n-2}}\mathrm{K}_{n-4,2}y^2 + \ldots),$$
$$1+A_{n,1}y \quad +A_{n,2}y^2 \quad + \ldots = \prod_{s_{n-1}}(1+\alpha_{n-1,s_{n-1}}\mathrm{K}_{n-3,1}y + \alpha^2_{n-1,s_{n-1}}\mathrm{K}_{n-3,2}y^2 + \ldots),$$

y being an undetermined quantity.

244. Multiplying out the right-hand sides of these identities, equating coefficients of like powers of y in each and employing the partition notation, we obtain

$$\mathrm{K}_{1,1} \quad = (1)_{\alpha_2}A_{1,1},$$
$$\mathrm{K}_{1,2} \quad = (2)_{\alpha_2}A_{1,2} + (1^2)_{\alpha_2}A^2_{1,1},$$
$$\mathrm{K}_{1,3} \quad = (3)_{\alpha_2}A_{1,3} + (21)_{\alpha_2}A_{1,2}A_{1,1} + (1^3)_{\alpha_2}A^3_{1,1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathrm{K}_{2,1} \quad = (1)_{\alpha_3}\mathrm{K}_{1,1},$$
$$\mathrm{K}_{2,2} \quad = (2)_{\alpha_3}\mathrm{K}_{1,2} + (1^2)_{\alpha_3}\mathrm{K}^2_{1,1},$$
$$\mathrm{K}_{2,3} \quad = (3)_{\alpha_3}\mathrm{K}_{1,3} + (21)_{\alpha_3}\mathrm{K}_{1,2}\mathrm{K}_{1,1} + (1^3)_{\alpha_3}\mathrm{K}^3_{1,1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathrm{K}_{n-3,1} = (1)_{\alpha_{n-2}}\mathrm{K}_{n-4,1},$$
$$\mathrm{K}_{n-3,2} = (2)_{\alpha_{n-2}}\mathrm{K}_{n-4,2} + (1^2)_{\alpha_{n-2}}\mathrm{K}^2_{n-4,1},$$
$$\mathrm{K}_{n-3,3} = (3)_{\alpha_{n-2}}\mathrm{K}_{n-4,3} + (21)_{\alpha_{n-2}}\mathrm{K}_{n-4,2}\mathrm{K}_{n-4,1} + (1^3)_{\alpha_{n-2}}\mathrm{K}^3_{n-4,1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_{n,1} \quad = (1)_{\alpha_{n-1}}\mathrm{K}_{n-3,1},$$
$$A_{n,2} \quad = (2)_{\alpha_{n-1}}\mathrm{K}_{n-3,2} + (1^2)_{\alpha_{n-1}}\mathrm{K}^2_{n-3,1},$$
$$A_{n,3} \quad = (3)_{\alpha_{n-1}}\mathrm{K}_{n-3,3} + (21)_{\alpha_{n-1}}\mathrm{K}_{n-3,2}\mathrm{K}_{n-3,1} + (1^3)_{\alpha_{n-1}}\mathrm{K}^3_{n-3,1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

245. We can now, by direct and successive substitution, express the quantities

$$A_{n,s}$$

in terms of the quantities

$$A_{1,s},$$

and the symmetric functions

$$(\ )_{a_2},\ (\ )_{a_3},\ \dots\ (\ )_{a_{n-1}},$$

and thence form directly any product

$$A_{n,s_1}A_{n,s_2}A_{n,s_3},\ \dots$$

expressed in the same manner.

246. We can also verify the relations

$$A_{n,1} = (1)_{a_2}(1)_{a_3}\dots(1)_{a_{n-1}}A_{1,1},$$

$$A^2_{n,1} - 2A_{n,2} = (2)_{a_2}(2)_{a_3}\dots(2)_{a_{n-1}}(A^2_{1,1} - 2A_{1,2}),$$

$$A^3_{n,1} - 3A_{n,2}A_{n,1} + 3A_{n,3} = (3)_{a_2}(3)_{a_3}\dots(3)_{a_{n-1}}(A^3_{1,1} - 3A_{1,2}A_{1,1} + 3A_{1,3}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$(m)_{a_n} = (m)_{a_2}(m)_{a_3}\dots(m)_{a_{n-1}}(m)_{a_1},$$

which have been previously established.

247. Reverting to the law of symmetry, which it is convenient to express in the form

$$A^{l_{n,1}}_{n,\lambda_{n,1}}A^{l_{n,2}}_{n,\lambda_{n,2}}\dots = \dots + L(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\dots)_{a_2}\dots(\lambda_{n-1,1}^{l_{n-1,1}}\lambda_{n-2,2}^{l_{n-2,2}}\dots)_{a_{n-1}}A^{l_{11}}_{1\lambda_{11}}A^{l_{12}}_{1\lambda_{12}}\dots + \dots,$$

where L is unchanged for any substitution impressed upon the $n - 1$ partitions

$$(\lambda_{s1}^{l_{s1}}\lambda_{s2}^{l_{s2}}\dots)$$
$$(s = 2, 3, \dots n);$$

I recall that in the present instance the partitions are restricted to contain as parts, positive, non-zero integers.

248. Guided by the distribution theorems of the first and second memoirs in this Journal, we may enquire the meaning of the number L in the theory of distributions and of the law of symmetry connected with it which is here brought forward.

§20.

249. In the first and second memoir I considered the distribution of objects of a certain type into parcels of a certain type and obtained a law of symmetry

by observing that it was immaterial whether an object was supposed attached to a parcel or a parcel attached to an object.

250. The consideration of both objects and parcels is in point of fact unnecessary. The parcels may be considered to be objects also, but of a different nature, and the distribution to be of objects of the first set with objects of the second set so as to form a number of pairs of objects, each pair consisting of an object from each set.

251. The result of this distribution may be regarded as the formation of a new set of objects of a two-fold character. Thus if the two sets of objects be

$$a_1,\ a_1,\ a_2,\ a_2, \text{ the first set,}$$
$$b_1,\ b_1,\ b_1,\ b_2, \text{ the second set,}$$

we may make a distribution

$$a_1b_1,\ a_1b_1,\ a_2b_1,\ a_2b_2,$$

and look upon this as a new set of four 2-fold objects. We may then say that we have distributed objects of type (2^2) with objects of type (31) so as to form two-fold objects of type (21^2).*

252. The new method of statement arises naturally from the observed reciprocity between parcels and objects. Together with the new set of two-fold objects we may now consider another set of objects, say

$$c_1,\ c_1,\ c_1,\ c_2,$$

and making any distribution

$$\begin{array}{cccc} a_1b_1, & a_1b_1, & a_2b_1, & a_2b_2, \\ c_1\ , & c_1\ , & c_1\ , & c_2\ , \end{array}$$

we arrive at a new set of three-fold objects

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2,$$

which constitutes objects (3-fold) of type (21^2) obtained by distributing objects (2-fold) of type (21^2) with objects of type (31).

253. Thus from the three sets of objects

$$a_1,\ a_1,\ a_2,\ a_2 \text{ of type } (2^2),$$
$$b_1,\ b_1,\ b_1,\ b_2 \text{ of type } (31),$$
$$c_1,\ c_1,\ c_1,\ c_2 \text{ of type } (31),$$

---

* In the first memoir this was stated to be a distribution of objects of type (2^2) into parcels of type (31) with a partition of restriction (21^2).

we have obtained a distribution

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2,$$

which constitutes objects (3-fold) of type (21^2).

254. Consider the problem of finding the number of distributions obtained from these three sets of objects, given by their types, so that a distribution may be constituted of 3-fold objects of given type (21^2).

255. The two sets of objects

$$\begin{gathered} a_1,\ a_1,\ a_2,\ a_2, \\ b_1,\ b_1,\ b_1,\ b_2, \end{gathered}$$

may be distributed into sets of two-fold objects of a variety of types. Each of these sets may be then distributed with the objects

$$c_1,\ c_1,\ c_1,\ c_2,$$

and one or more sets of three-fold objects of type (21^2) may or may not be thus reached.

256. Forming the relations

$$\begin{aligned} A_{3,1} &= (1)_2\mathrm{K}_{1,1}, \\ A_{3,3} &= (3)_2\mathrm{K}_{1,3} + (21)_2\mathrm{K}_{1,2}\mathrm{K}_{1,1} + (1^3)_2\mathrm{K}_{1,1}^3, \end{aligned}$$

we find

$$A_{3,3}A_{3,1} = \ldots. + 2(2^2)_2\mathrm{K}_{1,2}\mathrm{K}_{1,1}^2 + \ldots.,$$

which (see first memoir) shows that when objects of type (2^2) are distributed with objects of type (31), the set of two-fold objects formed is necessarily of type (21^2), and they are 2 in number.

257. We have now to find the number of distributions of objects of type (21^2) with objects of type (31) so that the distributions may be of type (21^2).

Writing

$$\begin{aligned} \mathrm{K}_{1,1} &= (1)_1T_1, \\ \mathrm{K}_{1,2} &= (2)_1T_2 + (1^2)_1T_1^2 \end{aligned}$$

we find

$$\mathrm{K}_{1,2}\mathrm{K}_{1,1}^2 = \ldots. + 2(31)_1T_2T_1^2 + \ldots.$$

From this it appears that any set of objects of type (21^2) may be distributed in two different ways with any set of objects of type (31) in such wise that the distribution is of type (21^2).

258. Combining the two results, we have

$$A_{3,3}A_{3,1} = \ldots. + 4(2^2)_2(31)_1T_2T_1^2 + \ldots.,$$

showing that the whole number of distributions, of the given type, of the three sets of objects is 4.

These are in fact

$$\begin{array}{l} a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2; \\ a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_2,\ a_2b_2c_1; \\ a_2b_1c_1,\ a_2b_1c_1,\ a_1b_1c_1,\ a_1b_2c_2; \\ a_2b_1c_1,\ a_2b_1c_1,\ a_1b_1c_2,\ a_1b_2c_1. \end{array}$$

259. Consider n sets of objects of types

$$P_1^{(1)},\ P_1^{(2)},\ \dots\dots P_1^{(r_1)} \qquad (r_1 = n)$$

and let it be demanded to find the number of distributions into n-fold objects of type

$$T.$$

260. The two sets of objects whose types are $P_1^{(1)}$, $P_1^{(2)}$ may be distributed into a set of two-fold objects of r_2 different types, say

$$P_{12}^{(1)},\ P_{12}^{(2)},\ \dots\dots P_{12}^{(r_2)}.$$

261. Selecting at pleasure any one of these sets of two-fold objects, say one of type P_{12}, we may distribute it with the set of objects of type $P_1^{(3)}$ and thus obtain a set of three-fold objects which may be of r_3 different types; say these are

$$P_{123}^{(1)},\ P_{123}^{(2)},\ \dots\dots P_{123}^{(r_3)}.$$

262. Again selecting one of these sets at pleasure, we may proceed successively until finally we arrive at a set of n-fold objects which may be of r_n different types which may be denoted by

$$P_{123\dots}^{(1)},\ P_{123\dots n}^{(2)},\ \dots\dots P_{123\dots n}^{(r_n)}.$$

One of these types may or may not be identical with the given type T.

263. Performing the process above indicated in all possible ways, we reach all the distributions into n-fold objects of the given type T.

264. The analytical process for arriving at the number of such distributions is simple and elegant. We have merely to combine the successive processes into a single process, as has been already done in the simple case considered which has gone before.

265. Recalling the previous notation and writing down the relation

$$A_{n,\lambda_{n,1}}^{l_{n,1}} A_{n,\lambda_{n,2}}^{l_{n,2}} \dots = \dots + L\,(\lambda_{21}^{l_{21}} \lambda_{22}^{l_{22}} \dots)_{a_2} \dots (\lambda_{n-1,1}^{l_{n-1,1}} \lambda_{n-2,2}^{l_{n-2,2}} \dots)_{a_{n-1}} A_{1\lambda_{11}}^{l_{11}} A_{1\lambda_{12}}^{l_{12}} \dots + \dots$$

I now state definitely the meaning to be attached to the number L which may be gathered from the preceding.

266. Let there be $n-1$ sets of objects of types

$$\begin{array}{l}(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots\ldots),\\ (\lambda_{31}^{l_{31}}\lambda_{32}^{l_{32}}\ldots\ldots),\\ \ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\\ (\lambda_{n,1}^{l_{n,1}}\lambda_{n,2}^{l_{n,2}}\ldots\ldots).\end{array}$$

The number of distributions into $n-1$-fold sets of objects of type

$$(\lambda_{11}^{l_{11}}\lambda_{12}^{l_{12}}\ldots\ldots)$$

is equal to L.

267. From this statement it is immediately obvious that any substitution whatever may be impressed upon the $n-1$ partitions

$$(\lambda_{s1}^{l_{s1}}\lambda_{s2}^{l_{s2}}\ldots\ldots);$$
$$(s=2,\ 3,\ \ldots\ldots\ n),$$

and hence in the above written identity the number L is unchanged when any substitution is impressed upon these $n-1$ partitions.

268. This distribution theorem thus involves an intuitive proof of the general law of symmetry.

269. It should be borne in mind that the general identity is constructed through the medium of the identities

$$\begin{array}{l}\mathrm{K}_{1,1}=(1)_{a_2}A_{1,1},\\ \mathrm{K}_{1,2}=(2)_{a_2}A_{1,2}+(1^2)_{a_2}A_{1,1}^2,\\ \mathrm{K}_{1,3}=(3)_{a_2}A_{1,3}+(21)_{a_2}A_{1,2}A_{1,1}+(1^3)_{a_2}A_{1,1}^3,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

$$s=2.\ 3,\ \ldots,\ (n-3)\left\{\begin{array}{l}\mathrm{K}_{s,1}=(1)_{a_{s+1}}\mathrm{K}_{s-1,1},\\ \mathrm{K}_{s,2}=(2)_{a_{s+1}}\mathrm{K}_{s-1,2}+(1^2)_{a_{s+1}}\mathrm{K}_{s-1,1}^2,\\ \mathrm{K}_{s,3}=(3)_{a_{s+1}}\mathrm{K}_{s-1,3}+(21)_{a_{s+1}}\mathrm{K}_{s-1,2}\mathrm{K}_{s-1,1}+(1^3)_{a_{s+1}}\mathrm{K}_{s-1,1}^3,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right.$$

$$\begin{array}{l}A_{n,1}=(1)_{a_{n-1}}\mathrm{K}_{n-3,1},\\ A_{n,2}=(2)_{a_{n-1}}\mathrm{K}_{n-3,2}+(1^2)_{a_{n-1}}\mathrm{K}_{n-3,1}^2,\\ A_{n,3}=(3)_{a_{n-1}}\mathrm{K}_{n-3,3}+(21)_{a_{n-1}}\mathrm{K}_{n-3,2}\mathrm{K}_{n-3,1}+(1^3)_{a_{n-1}}\mathrm{K}_{n-3,1}^3\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

270. I propose to obtain the general result for the case of $n = 4$ and three objects in each set. There are then three sets of objects.

We have

$$\begin{aligned}
\mathrm{K}_{1,1} &= (1)_{\alpha_2} A_{1,1}, \\
\mathrm{K}_{1,2} &= (2)_{\alpha_2} A_{1,2} + (1^2)_{\alpha_2} A_{1,1}^2, \\
\mathrm{K}_{1,3} &= (3)_{\alpha_2} A_{1,3} + (21)_{\alpha_2} A_{1,2} A_{1,1} + (1^3)_{\alpha_2} A_{1,1}^3; \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A_{4,1} &= (1)_{\alpha_3} \mathrm{K}_{1,1}, \\
A_{4,2} &= (2)_{\alpha_3} \mathrm{K}_{1,2} + (1^2)_{\alpha_3} \mathrm{K}_{1,1}^2, \\
A_{4,3} &= (3)_{\alpha_3} \mathrm{K}_{1,3} + (21)_{\alpha_3} \mathrm{K}_{1,2} \mathrm{K}_{1,1} + (1^3)_{\alpha_3} \mathrm{K}_{1,1}^3, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

in order to eliminate the quantities K.

271. The result is, writing $(3)_{\alpha_s} = (3)_s$ for brevity, and so on,

$$\begin{aligned}
A_{4,3} = {} & (3)_3(3)_2 A_{1,3} + \{(3)_3(21)_2 + (21)_3(3)_2\} A_{1,2} A_{1,1} + \{(3)_3(1^3)_2 \\
& + (1^3)_3(3)_2\} A_{1,1}^3 + (21)_3(21)_2 (A_{12} A_{11} + A_{11}^3) \\
& + 3\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{1,1}^3 + 6(1^3)_3(1^3)_2 A_{1,1}^3, \\
A_{4,2} A_{4,1} = {} & (3)_3(3)_2 A_{12} A_{11} + \{(3)_3(21)_2 + (21)_3(3)_2\}(A_{12} A_{11} + A_{11}^3) \\
& + 3\{(3)_3(1^3)_2 + (1^3)_3(3)_2\} A_{11}^3 + (21)_3(21)_2 (A_{12} A_{11} + 4A_{11}^3) \\
& + 9\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{11}^3 + 18(1^3)_3(1^3)_2 A_{11}^3, \\
A_{4,1}^3 = {} & (3)_3(3)_2 A_{11}^3 + 3\{(3)_3(21)_2 + (21)_3(3)_2\} A_{11}^3 + 6\{(3)_3(1^3)_2 + (1^3)_3(3)_2\} A_{11}^3 \\
& + 9(21)_3(21)_2 A_{11}^3 \qquad + 18\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{11}^3 \\
& \qquad + 36(1^3)_3(1^3)_2 A_{11}^3.
\end{aligned}$$

272. The symmetry is manifest, and representing by

$$[(3)(21)(1^3)]$$

a distribution of three sets of objects of types (3), (21) and (1^3) and so forth, we obtain for the numbers of the different types

$$\begin{aligned}
[(3)\,,\ (3)\,,\ (3)\,] &= A_{1,3}, \\
[(3)\,,\ (3)\,,\ (21)] &= A_{1,2} A_{1,1}, \\
[(3)\,,\ (3)\,,\ (1^3)\,] &= A_{1,1}^3, \\
[(3)\,,\ (21),\ (21)] &= A_{1,2} A_{1,1} + A_{1,1}^3, \\
[(3)\,,\ (21),\ (1^3)\,] &= 3A_{1,1}^3, \\
[(3)\,,\ (1^3)\,,\ (1^3)\,] &= 6A_{1,1}^3, \\
[(21),\ (21),\ (21)] &= A_{1,2} A_{1,1} + 4A_{1,1}^3, \\
[(21),\ (21),\ (1^3)\,] &= 9A_{1,1}^3, \\
[(21),\ (1^3)\,,\ (1^3)\,] &= 18A_{1,1}^3, \\
[(1^3)\,,\ (1^3)\,,\ (1^3)\,] &= 36A_{1,1}^3,
\end{aligned}$$

where for example the seventh of these results is to be interpreted as indicating that of three sets of objects of types (21), (21) and (21) respectively, there is one distribution of type (21) and four of type (1^3).

273. Taking the objects to be

$$\begin{array}{l} a_1,\ a_1,\ a_2 \text{ of type } (21), \\ b_1,\ b_1,\ b_2 \text{ of type } (21), \\ c_1,\ c_1,\ c_2 \text{ of type } (21), \end{array}$$

the five distributions are in fact

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_2c_2 \text{ of type } (21)$$

and

$$\left.\begin{array}{l} a_1b_1c_1,\ a_1b_1c_2,\ a_2b_2c_1 \\ a_1b_1c_1,\ a_1b_2c_2,\ a_2b_1c_1 \\ a_1b_2c_1,\ a_1b_1c_1,\ a_2b_1c_2 \\ a_1b_1c_2,\ a_1b_2c_1,\ a_2b_1c_1 \end{array}\right\} \text{ of type } (1^3),$$

and observe that there are no others.

274. I have extended the subject of these four memoirs in a paper under the title "Memoir on Symmetric Functions of the Roots of Systems of Equations" in the Philosophical Transactions of the Royal Society of London, Vol. 181 (1890), A, pp. 481–536.

Woolwich, *January* 10, 1890.

Multivalent and Univalent Involutory Correspondences in a Plane determined by a Net of Curves of n^{th} Order.

A paper read before the New York Mathematical Society at the Meeting of February 6th, 1891.

BY CHAS. PROTEUS STEINMETZ, *Yonkers, N. Y.*

INTRODUCTION.

[In the following, points will always be denoted by German letters, lines and curves by Latin letters; planes, pencils and nets by Greek letters.]

§1.—*Ranges of Points.*

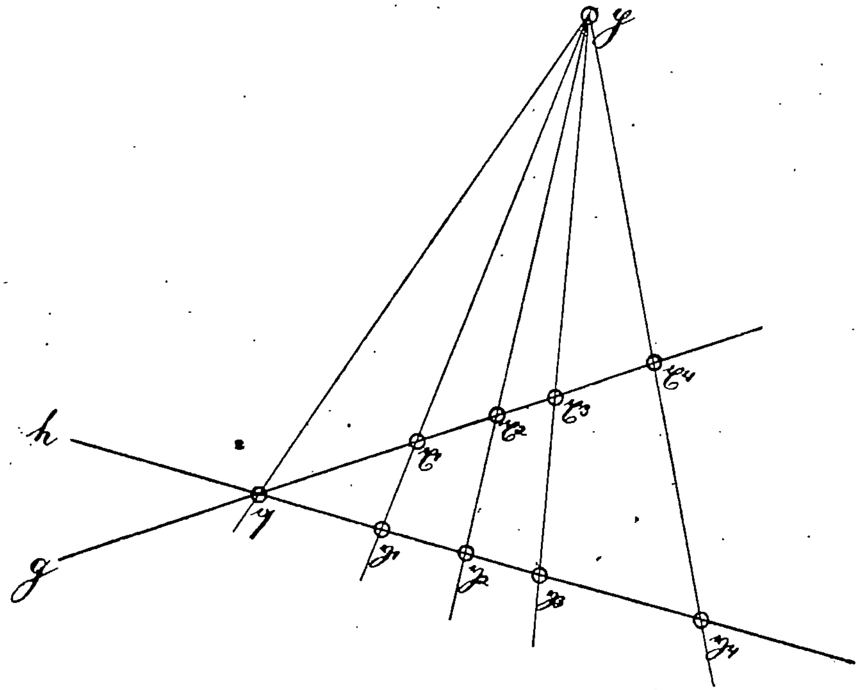


FIG. 1.

Let us assume two straight lines g and h. Then, between the points $\mathfrak{x}$ and

$\mathfrak{y}$ of these lines g and h, a correspondence can be established so that to any point $\mathfrak{x}$ on g corresponds a point $\mathfrak{y}$ on h.

The simplest way to produce such a correspondence is from a point $\mathfrak{p}$ taken at random, but outside of g and h, to draw a pencil of rays and attribute to each other the two points of intersection of any ray with the two lines g and h.

The correspondence established in this way is called "*perspective*," with the point $\mathfrak{p}$ as the "*centre of perspectivity*," and has the essential property that the *anharmonic ratio* of any four points $\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4$ is the same as the anharmonic ratio of their corresponding points $\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4$:

$$(\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4) \barwedge (\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4),$$

or

$$\frac{\mathfrak{x}_1\mathfrak{x}_3}{\mathfrak{x}_2\mathfrak{x}_3} \div \frac{\mathfrak{x}_1\mathfrak{x}_4}{\mathfrak{x}_2\mathfrak{x}_4} = \frac{\mathfrak{y}_1\mathfrak{y}_3}{\mathfrak{y}_2\mathfrak{y}_3} \div \frac{\mathfrak{y}_1\mathfrak{y}_4}{\mathfrak{y}_2\mathfrak{y}_4}.$$

At the point of intersection $\mathfrak{q}$ of the lines g and h lie two corresponding points, one in each range.

Analytically this correspondence is represented by a bilinear equation between the coordinates x and y of the points $\mathfrak{x}$ and $\mathfrak{y}$ of the ranges or forms g and h.

But if we reverse this reasoning and define the correspondence between the two ranges g and h by the condition that the anharmonic ratio of any four points $\mathfrak{x}_1\mathfrak{x}_2\mathfrak{x}_3\mathfrak{x}_4$ is the same as the anharmonic ratio of their corresponding points $\mathfrak{y}_1\mathfrak{y}_2\mathfrak{y}_3\mathfrak{y}_4$, then in general the connecting lines of corresponding points $\mathfrak{x}$ and $\mathfrak{y}$ do not intersect in a centre of perspectivity $\mathfrak{p}$, but envelop a conic curve, and the correspondence is called *projective* or *homographic*, so that perspectivity appears as a special case of projectivity.

But even two general projective ranges of points can be brought to perspectivity by merely moving them so that the point of intersection $\mathfrak{q}$ of the ranges g and h corresponds with itself.

Now, if these two ranges g and h coincide with the same line, then by the relation of projectivity a correspondence between the points of this line—considered as double line—is established so that to a point $\mathfrak{x}$, considered as a point of g, corresponds a point $\mathfrak{y}$ of h, and to this point $\mathfrak{y}$, considered as a point on g, corresponds again a point $\mathfrak{z}$ on h.

If, now, this point $\mathfrak{z}$ is again the same point $\mathfrak{x}$, so that to a point $\mathfrak{x}$, considered as point on g, corresponds on h the same point $\mathfrak{y}$, as to $\mathfrak{x}$, considered as point on h, corresponds on g, then this pair of points $\mathfrak{x}$, $\mathfrak{y}$ is said to be in *involution*, and it can be proved that when in two coincident projective ranges of points one pair of corresponding points is in involution, every pair of corresponding points $\mathfrak{x}$ and $\mathfrak{y}$ is in involution, and the correspondence between these coincident projective ranges of points is called an *involutory correspondence*, or in short an *involution.*

Then we see the individualities of those two coincident projective ranges of points g and h disappear, and the quadratic involution can be considered as such a combination or configuration of the points of one range, that to any point $\mathfrak{x}$ corresponds a point $\mathfrak{y}$, and to point $\mathfrak{y}$ again the point $\mathfrak{x}$.

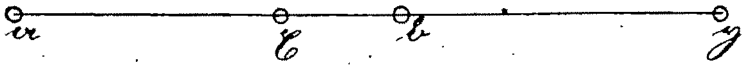


FIG. 2.

Such an involution is formed for instance by the third and fourth harmonic point. Because, when you assume 2 points $\mathfrak{a}$ and $\mathfrak{b}$, to a point $\mathfrak{x}$ always corresponds a point $\mathfrak{y}$ as fourth harmonic point with respect to $\mathfrak{a}$, $\mathfrak{b}$, and to $\mathfrak{y}$ corresponds as fourth harmonic point again the point $\mathfrak{x}$.

The points $\mathfrak{a}$ and $\mathfrak{b}$ are the *self-corresponding points* or *centre-points* of this involutory correspondence.

This harmonic range is a particular case of the general quadratic involution: the involution with real centre-points.

We call this involution a *univalent correspondence*, because to a point $\mathfrak{x}$ corresponds always one point $\mathfrak{y}$.

Analytically it is given by the roots of a quadratic equation, the coefficients of which are linear functions of a variable parameter.

A *multivalent involution* on a range of points will be such a correspondence that to one point $\mathfrak{x}$ corresponds a certain number of points $\mathfrak{y}_1, \mathfrak{y}_2 \dots \mathfrak{y}_{n-1}$, and to every point $\mathfrak{y}$ corresponds again the point $\mathfrak{x}$ and all the other points $\mathfrak{y}$, so that the points of the range are joined into groups of n, so that to a point $\mathfrak{x}$ of a group correspond the other $(n-1)$ points of the same group, and inversely.

Such a correspondence is, for instance, produced by a linear pencil of curves of n^{th} order. Through any point $\mathfrak{x}$ of g passes one, and only one, curve of the pencil which intersects g in $(n-1)$ additional points $\mathfrak{y}$ corresponding to $\mathfrak{x}$, and to any one of those points $\mathfrak{y}$ correspond again the other points $\mathfrak{y}$ together with $\mathfrak{x}$.

Therefore this correspondence is $(n-1)$-valent.

It contains $2(n-1)$ centre-points, or self-corresponding points, because $2(n-1)$ curves of the pencil of curves of n^{th} order touch the line g.

For further particulars on involutions in ranges of points see Weyr, Wiener Berichte, 1879.

§2.—*Planes in Correspondence.*

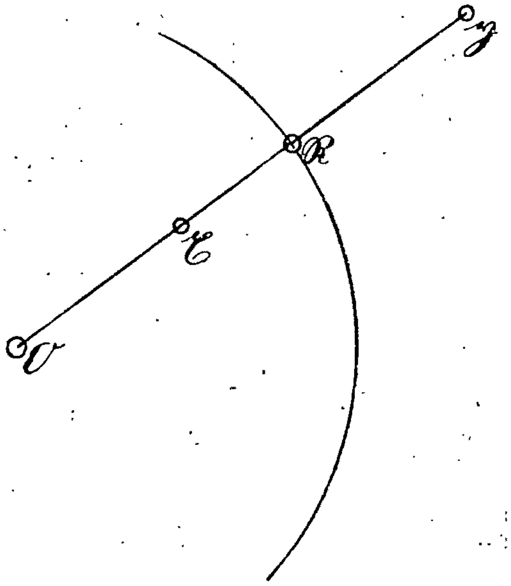


FIG. 8.

In the same way as between ranges of points, between two planes ε and η a correspondence of points, can be established so that to every point $\mathfrak{x}$ on ε corresponds one, or several points $\mathfrak{y}$ on η, and this correspondence can therefore be either *univalent* or *multivalent.*

If, now, the two planes ε and η coincide, to a point $\mathfrak{x}$, considered as point on ε, corresponds a point $\mathfrak{y}$ on η, and to $\mathfrak{y}$, considered as point on ε, corresponds again a point $\mathfrak{z}$ on η, and if this point $\mathfrak{z}$ is always the same as $\mathfrak{x}$, that is, if to a point $\mathfrak{x}$ as point on ε corresponds on η the same point $\mathfrak{y}$, as to $\mathfrak{x}$, considered as point on η, corresponds in ε, then the two coincident planes are said to be in *involutory coincidence* and produce an *involutory correspondence*; and again the individualities of the two different planes disappear.

Such an involutory correspondence of elementary character is, for instance, the correspondence by reciprocal radii.

Let us assume (Fig. 3) a point $\mathfrak{D}$ as centre. Then to every point $\mathfrak{x}$ is attributed on the ray $|\mathfrak{D}\mathfrak{x}|$ a point $\mathfrak{y}$ by the condition

$$|\mathfrak{D}\mathfrak{y}| = \frac{r^2}{|\mathfrak{D}\mathfrak{x}|},$$

or

$$|\mathfrak{D}\mathfrak{x}| \times |\mathfrak{D}\mathfrak{y}| = r^2,$$

where r^2 is a given quantity; and to $\mathfrak{y}$ corresponds again the point $\mathfrak{x}$.

All the points corresponding with themselves lie upon a circle with $\mathfrak{D}$ as centre and r as radius, which is called the *centre-curve* or *self-corresponding curve* of the correspondence; and this correspondence is called quadratic or of the 2^{d} order, because to the points of a straight line corresponds a circle, that is, a curve of 2^{d} order.

In this correspondence by reciprocal radii the points of the plane are joined into groups of two, and the connecting lines of corresponding points all going through the point $\mathfrak{D}$, the correspondence is *perspective*, with point $\mathfrak{D}$ as the *centre of perspectivity*.

Again, a *multivalent involutory correspondence* in the plane is a correspondence in which to a point $\mathfrak{x}$ belong a certain number of points $\mathfrak{y}_1, \mathfrak{y}_2 \ldots . \mathfrak{y}_n$, and to any one of these points $\mathfrak{y}$ correspond the other points $\mathfrak{y}$, together with point $\mathfrak{x}$.

This multivalent involutory correspondence is $(n-1)$-valent, and joins the points of the plane into groups of n.

It is these multivalent and univalent involutory correspondences in the plane which I propose to consider here, the corresponding relations in spaces having been published in the "Zeitschrift für Mathematik und Physik," Dresden, 1890, p. 219.

CHAPTER I.

Correspondences defined by a General Net.

§3.—*Definition and Quantivalence of the Correspondence.*

Let us assume in the plane a general net of curves of n^{th} order; that is, all those curves which, from three curves given at random—but which we first suppose to have no common point of intersection—are produced linearly, that is, by continued formation of pencils of curves.

Analytically, this net of curves is represented by the equation

$$F(x, y) \equiv \theta_1 f_1(x, y) + \theta_2 f_2(x, y) + \theta_3 f_3(x, y) = 0.$$

Then all the curves c of this net K, which intersect in one point $\mathfrak{x}$, form a pencil of curves of n^{th} order, and therefore intersect in $(n^2 - 1)$ additional points $\mathfrak{y}$, which, together with $\mathfrak{x}$, are the base-points of this pencil.

Therefore, to every point $\mathfrak{x}$ correspond $(n^2 - 1)$ points $\mathfrak{y}$.

On the other hand, to every one of those points $\mathfrak{y}$ correspond the point $\mathfrak{x}$ and the other $(n^2 - 2)$ points $\mathfrak{y}$.

Hence these points $\mathfrak{x} \ldots\ldots \mathfrak{y}$ define an involutory

$$N = (n^2 - 1)\text{-valent}$$

correspondence of the points of the plane.

The locus of all those points $\mathfrak{x}$ which coincide with one of their corresponding points $\mathfrak{y}$, and thus correspond with themselves, is a curve of the $3(n-1)^{\text{th}}$ order $H^{3(n-1)}$, the *centre-curve* or *self-corresponding curve* of the correspondence. (Proof, see §6.)

§4.—*The Order of the Correspondence.*

To determine the order of the correspondence, we produce the curve C which corresponds to the straight line g.

For this purpose we make use of a representation of the curves c of n^{th} order of the net K by the straight lines l of the plane E, in attributing to each curve of n^{th} order c its linear or last polar-line with regard to a fixed point $\mathfrak{P}$ of the plane.*

Then to each line l of E corresponds a curve c of K, and to each curve c of K corresponds a straight line l of E.

To each pencil of curves c in K corresponds a pencil of rays in E, and inversely.

Hence to every point $\mathfrak{p}$ of K corresponds in E one point $\mathfrak{q}$, the base-point of the pencil of lines l in E, which corresponds to that pencil of curves c in K, which is determined by $\mathfrak{p}$ as one of its base-points.

---

* But which point $\mathfrak{P}$ is supposed to be no particular point of the net K.

But, to every point $\mathfrak{q}$ in E correspond n^2 points $\mathfrak{p}$ in K, the base-points of the pencil of curves c corresponding to the pencil of lines with $\mathfrak{q}$ as centre.

This representation is therefore $1 \div n^2$-valent.

To a straight line g in K corresponds in E a curve k, which cuts any line l in the same number of points $\mathfrak{q}$, as g cuts the curve c, corresponding in K to the line l in E, that is, in n points.

Hence k is a (unicursal) curve of the n^{th} order.

To this curve $k^{(n)}$ corresponds in K a curve of the n^2th order, which consists of the line g and a curve $C^{(n^2-1)}$ of the $(n^2-1)^{\text{th}}$ order, the curve corresponding to g. That is—

"To every straight line g corresponds a curve $C^{(n^2-1)}$ of the order

$$M = (n^2 - 1),$$

as the locus of the groups of (n^2-1) points $\mathfrak{y}$, corresponding to the points $\mathfrak{x}$ of the line g."

"The order of this involutory multivalent correspondence is: $M = (n^2-1)$; that is, the same as the quantivalence."*

§5.—*Points and Lines in the Correspondence.*

Each line g intersects the corresponding curve $C^{(n^2-1)}$ in (n^2-1) points, of which

$3(n-1)$ points are centre-points or self-corresponding points, as points of intersection with the centre-curve $H^{3(n-1)}$.

The other $(n^2-1) - 3(n-1) = (n-1)(n+2)$ points correspond to each other mutually:

"On every line there are $\frac{(n-1)(n+2)}{2}$ pairs of *conjugate* or *mutually corresponding points*."

If a line g_0 contains more than $\frac{(n-1)(n+2)}{2}$ pairs of conjugate or mutually corresponding points, it corresponds to itself; that is, all the points of the line g_0

---

* But these values of the order, and of the quantivalence of the correspondence, are derived only under the assumption of the non-existence of fundamental points, or common base-points, in the net, and do not hold if all the curves c of the net K contain common points of intersection. For this case see Chapter III.

correspond to one another, and the corresponding curve of g_0 is of the order (n^2-2) only, or even of lower order.

"On two lines g_1 and g_2, given at random, there are (n^2-1) pairs of corresponding points."

For g_2 intersects $C_1^{(n^2-1)}$ in (n^2-1) points.

"On every line g there are $3(n-1)(n^2-2)$ points which have a centre-point as a corresponding point."

For $C^{(n^2-1)}$ intersects $H^{3(n-1)}$ in $3(n-1)(n^2-1)$ points, of which $3(n-1)$ points lie on g.

Hence

"All the points $\mathfrak{x}$, which have a centre-point as corresponding point, produce a curve $K^{3(n-1)(n^2-2)}$ of the $3(n-1)(n^2-2)$th order, which I call the *conjugate curve to the centre-curve* $H^{3(n-1)}$, because $K^{3(n-1)(n^2-2)}$ together with $H^{3(n-1)}$ corresponds to $H^{3(n-1)}$."

§6.—*The Centre-curve* $H^{3(n-1)}$ *of the Correspondence.*

Let $\mathfrak{z}$ be a centre-point, that is, a point which corresponds to itself, or rather to a point $\mathfrak{z}_1$ infinitely near to $\mathfrak{z}$. Then the line $t = |\mathfrak{z}\mathfrak{z}_1|$ is common tangent of all the curves c of the pencil determined by $\mathfrak{z}$.

Hence the centre-curve is the locus of contact-points of all the pencils with a common tangent, or of those points where all the curves of a pencil touch each other.

In such a pencil of tangent curves c, always one curve d is determined by a point $\mathfrak{z}_2$ infinitely near to the contact-point $\mathfrak{z}$, but outside of the common tangent $t = |\mathfrak{z}\mathfrak{z}_1|$.

This curve d has a node or double point in $\mathfrak{z}$.

Hence the centre-curve is the locus of the nodes of the curves c of the net K.

Now the first polar-curve of any point of the plane with respect to a given curve passes through all the nodes of this curve.

In a centre-point intersect therefore the first polar-curves of all the points of the plane, or what is the same, of three points not being in a straight line, with respect to that curve d of the net K, which has a node in this centre-point.

Now the first polar-curves of the points of the plane, taken with respect to the curves of nth order c of the net K, produce projective nets of curves of $(n-1)$th order, which are projective also to the net K.

Hence the centre-curve H is the product of 3 projective nets of curves of $(n-1)^{th}$ order, which are the first polar-curves of 3 points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, given at random, but so that they do not lie in a straight line, with respect to the curves of n^{th} order c of the net K.

Let these three projective nets be called A, B, Γ, their curves a, b, c.

Assume at random a straight line g. Every point $\mathfrak{p}$ of this straight line g determines a pencil α of curves a in net A, to which corresponds in B a pencil β of curves b. One of these curves b passes through $\mathfrak{p}$, and to this curve b corresponds in net Γ one curve c, which intersects g in $(n-1)$ points $\mathfrak{q}$, so that to a point $\mathfrak{p}$ correspond $(n-1)$ points $\mathfrak{q}$.

On the other hand, every point $\mathfrak{q}$ on g determines a pencil γ of curves c of $(n-1)^{th}$ order of the net Γ, to which correspond two projective pencils α and β of curves a and b of the nets A and B.

These pencils of curves α and β, of $(n-1)^{th}$ order, produce by their projective correspondence a curve of $2(n-1)^{th}$ order, which intersects g in $2(n-1)$ points $\mathfrak{p}$ corresponding to $\mathfrak{q}$. Thus:

to every point $\mathfrak{p}$ correspond $(n-1)$ points $\mathfrak{q}$;
to every point $\mathfrak{q}$ correspond $2(n-1)$ points $\mathfrak{p}$;

hence the coincident ranges of points $\mathfrak{p}$ and $\mathfrak{q}$ on line g have $3(n-1)$ double-points or self-corresponding points,* which, as points of intersection of 3 corresponding first polar-curves a, b, c, are points of the centre-curve H. That is,

"The centre-curve $H^{3(n-1)}$ of the correspondence is of the $3(n-1)^{th}$ order."

CHAPTER II.

CORRESPONDENCES DEFINED BY A NET WITH FIXED BASE-POINTS OR FUNDAMENTAL POINTS.

§7.—*Definition and Quantivalence.*

If all the curves c of n^{th} order of the net K pass through the same point $\mathfrak{P}$, there intersecting each other, we call this point $\mathfrak{P}$ a (*simple*) *fundamental point* of the correspondence.

---

* In general, μ-projective $(\mu-1)$-dimensional linear systems of curves of m^{th} order produce as the locus of their common points of intersection of corresponding curves a curve of the order μm, and therefore, for $m=n-1$, $\mu=3$, $3(n-1)$.

If all the curves c pass through the same point $\mathfrak{O}$, and have this point as a κ-ple point, we call this point $\mathfrak{O}$ a *fundamental point of the correspondence of multiplicity* κ, or a *κ-ple fundamental point.*

At a simple fundamental point $\mathfrak{P}$, one curve c has a node or double-point, but in general, at a κ-ple fundamental point no curve c has a multiple-point of order higher than κ.

Therefore all fundamental points are points of the centre-curve H.

Every κ-ple fundamental point is a κ^2-ple point of intersection of the curves c of the net K, and therefore consumes κ^2 points $\mathfrak{y}$, corresponding to every point $\mathfrak{x}$ of the plane. Hence

"The quantivalence of the correspondence is

$$N = n^2 - \Sigma\kappa_i^2 - 1,$$

where κ_i is the multiplicity of the fundamental point $\mathfrak{O}_i$."

§8.—*Singular Curves and Singular Points.*

"Of fundamental points of a higher multiplicity than $n/2$, only one can exist. Then all the other fundamental points must be of a multiplicity equal to or less than $n - \kappa$, where κ is the multiplicity of the first point."

"If the fundamental points $\mathfrak{O}_1, \mathfrak{O}_2 \ldots . \mathfrak{O}_i$ of the multiplicities $\kappa_1, \kappa_2 \ldots . \kappa_i$ lie upon a curve $S^{(m)}$ of the m^{th} order, and have for this curve $S^{(m)}$ the multiplicities $\theta_1, \theta_2 \ldots . \theta_i$, we must have

$$\Sigma\kappa_i\theta_i \leqq mn."$$

Otherwise this curve $S^{(m)}$ would be a part of all the curves c of the net K, and this net K therefore would reduce to a net of $(n - m)^{\text{th}}$ order.

A special case is—

"The sum of the multiplicities of all the fundamental points lying on a straight line must be equal to or less than n."*

---

* In the corresponding relations in space these conditions do not exist, but fundamental curves and fundamental lines, that is, curves and lines common to all the surfaces of n^{th} order of the tri-dimensional system which defines the correspondence, can exist, and indeed must exist, if the correspondence is univalent.

"If the fundamental points $\mathfrak{O}_i$ of the multiplicities $\varkappa_i$ lie on a curve $S^{(m)}$ of the m^{th} order, and are for this curve of the multiplicity θ_i, and if

$$\Sigma \varkappa_i \theta_i = mn,$$

we call this curve $S^{(m)}$ a *singular curve of m^{th} order of the correspondence.*"

The pencil of curves c, determined by a point of a singular curve of m^{th} order $S^{(m)}$, consists of this curve $S^{(m)}$ and of a pencil of curves of $(n-m)^{th}$ order which intersect in

$$(n-m)^2 - \Sigma \varkappa_i'^2 - \Sigma (\varkappa_i'' - \theta_i)^2$$

points (where $\varkappa_i'$ is the multiplicity of those fundamental points $\mathfrak{O}_i'$ which do not lie on $S^{(m)}$, and $\varkappa_i''$ the multiplicity of those fundamental points $\mathfrak{O}_i''$ which are θ_i-ple points for $S^{(m)}$). These points we term the *singular points* $\mathfrak{S}$, *associated with the singular curve* $S^{(m)}$.

To any point of a singular curve $S^{(m)}$ corresponds the whole singular curve $S^{(m)}$ and their associated singular points $\mathfrak{S}$.

To a singular point $\mathfrak{S}$ correspond its other associated singular points and the whole associated singular curve $S^{(m)}$.

Therefore every singular curve $S^{(m)}$ is a part of the centre-curve H of the correspondence, but not in general any singular point.

Hence the order of the centre-curve H is

$$3(n-1) - \Sigma m_i,$$

and

"The sum of the orders of the singular curves $S^{(m)}$ is equal to or less than $3(n-1)$,

$$\Sigma m_i \leqq 3(n-1)."$$

§9.—*The Order of the Correspondence.*

The fundamental points have no influence upon the order of the correspondence, and, if no singular curves exist, the order of the correspondence is

$$M = n^2 - 1.$$

Every line g intersects a singular curve $S^{(m)}$ of m^{th} order in m points, to every one of which the whole curve $S^{(m)}$ corresponds, together with its singular points.

Hence, if a singular curve of m^{th} order exists, the curve corresponding to a straight line g consists of the curve $S^{(m)}$, counted m-fold, and a curve C of the $(n^2 - m^2 - 1)^{\text{th}}$ order. Thus:

"The order of the correspondence is

$$M = n^2 - \Sigma m_i^2 - 1,$$

where m_i is the order of the singular curves $S^{(m)}$."

§10.—*The General Curve C.*

"To a general straight line g corresponds a curve C of the

$$M = (n^2 - \Sigma m_i^2 - 1)^{\text{th}} \text{ order},$$

which has every $\varkappa_i$-ple fundamental point $\mathfrak{D}_i$ as a

$$\nu = n\varkappa_i\text{-ple point},$$

if $\mathfrak{D}_i$ does not lie on a singular curve, but only as a

$$\nu' = (n\varkappa_i' - \Sigma m_\sigma \theta_\sigma)\text{-ple point},$$

if $\mathfrak{D}_i'$ lies on the singular curves $S_\sigma^{(m)}$ and is a θ_σ-ple point for those curves."

This curve C has every singular point $\mathfrak{S}$, which is associated with a singular curve $S^{(m)}$ of m^{th} order, as a

$$\nu'' = m\text{-ple point},$$

if $\mathfrak{S}$ does not lie on any singular curve, but only as a

$$\nu''' = (m - \Sigma m_\sigma \theta_\sigma)\text{-ple point},$$

if $\mathfrak{S}$ is a θ_σ-ple point of the singular curve $S^{(m)}$.

Between these numbers ν, ν', ν'', ν''', the quantivalence N and the order M exists the relation

$$\Sigma(\nu^2 + \nu'^2 + \nu''^2 + \nu'''^2) = (M + 1)(M - N).$$

To prove these results, we make use of a construction for the curve C different from that given in §4.

We produce the curve C, corresponding to a straight line g, by two pencils α and β of curves a and b of the net K, which pencils a and b, as belonging to the same net, have one curve c in common.

These curves a and b have with each other an n-valent correspondence by means of the points of the line g; to every curve a corresponding to the n-curves b, which are determined by the points of intersection of a with g.

Now these n-curves b intersect a in n^3 points. n of these are points of the line g, and the other $n^3 - n = n(n^2 - 1)$ points are the points of intersection of the curve a with the curve C, and, a being of the n^{th} order, C therefore is of the $(n^2 - 1)^{th}$ order.

If $S^{(m)}$ is a singular curve of m^{th} order, it intersects a in nm points. Each one of these is a m-ple point of the locus corresponding to g, because corresponding to m points of g, and the real curve C has therefore only $n(n^2 - m - 1)$ points of intersection with a; that is, C is of the $(n^2 - m - 1)^{th}$ order.

Now, let $\mathfrak{Q}_i$ be a fundamental point of multiplicity $\varkappa_i$, then it is $\varkappa$-ple point of intersection of a curve a with any curve b. Therefore it is a $n\varkappa_i^2$-ple point of intersection of a with the n corresponding curves b, that is, with the curve C, and, $\mathfrak{Q}_i$ being $\varkappa_i$-ple point of a, it is $n\varkappa_i$-ple point of C. But if it is a θ-ple point of a singular curve $S^{(m)}$ of m^{th} order, it is $m\theta\varkappa_i$-ple point of intersection of a with the curve $S^{(m)}$, counted m-fold, and therefore only a $(n\varkappa_i^2 - m\theta\varkappa_i)$-ple point of intersection of a with C, that is, a $(n\varkappa_i - m\theta)$-ple point of C, etc.

§11.—*Particular Curves C.*

If a line g passes through a ρ-ple fundamental point $\mathfrak{R}$, the pencils a and b are put in a correspondence only $(n - \rho)$-valent, because every curve a intersects g in only $(n - \rho)$ variable points. If g passes through a singular point $\mathfrak{W}$ associated with a singular curve of w^{th} order, $S^{(w)}$, this curve $S^{(w)}$ disintegrates from the curve C corresponding to g, and in considering the number of points of intersection coinciding with a fundamental point, or with a singular point, we get in the same way as in §10:

"To a straight line g, passing through the fundamental points $\mathfrak{R}_\mu$ of $\rho_\mu{}^{th}$ multiplicity and the singular points $\mathfrak{W}_\tau$, which are associated with the singular curves $S_\tau^{(w)}$ of $w_\tau{}^{th}$ order, corresponds a curve C of the

$$M = (n(n - \Sigma\rho_\mu) - \Sigma m_\iota(m_\iota - \theta_i) - \Sigma w_\tau - 1)^{th} \text{ order,}$$

where θ_ι is the multiplicity of $\mathfrak{R}_\mu$ for S_ι. And this curve, at a fundamental point $\mathfrak{Q}_i$ of the multiplicity $\varkappa_\iota$, which fundamental point is θ_σ-ple point for the singular curves $S_\sigma^{(m)}$, and θ_τ-ple point for the singular curves $S_\tau^{(w)}$, has a

$$\nu = ((n - \Sigma\rho_\mu)\varkappa_i - \Sigma_\sigma m\theta_\sigma - \Sigma\theta_\tau)\text{-ple point;}$$

at a point $\mathfrak{R}_\mu$ it has a

$$\nu' = ((n - \Sigma\rho_\mu)\rho_\mu - \Sigma m_\sigma\theta_\sigma - \Sigma\theta_\tau - 1)\text{-ple point;}$$

at a singular point $\mathfrak{S}_i$ it has a

$$\nu'' = (m_i - \Sigma m_\sigma \theta_\sigma)\text{-ple point,}$$

where m_i is the order of the singular curve $S_i^{(m)}$, associated with the singular point $\mathfrak{S}_i$; which is a θ_σ-ple point for the singular curve $S_\sigma^{(m)}$; finally, at a singular point $\mathfrak{S}_{\mathfrak{w}}$, associated with a point $\mathfrak{W}_\tau$, C has a

$$\nu''' = (w_\tau - \Sigma m_\sigma \theta_\sigma + 1)\text{-ple point.}$$

Between these numbers ν, ν', ν'', ν''' and ν_1, ν_1', ν_1'', ν_1''' of two curves C and C_1 of orders M and M_1 respectively, and the quantivalence N, exists the relation

$$\begin{aligned}\Sigma\,(\nu\nu_1 + \nu'\nu_1' + \nu''\nu_1'' + \nu'''\nu_1''') + N \\ = MM_1 - (M - s)(N - 1) = MM_1 - (M_1 - s_1)(N - 1),\end{aligned}$$

where s is the sum of the multiplicities of the fundamental and singular points on g_1 for C."

As a particular case of this theorem, we have—

"To a straight line s, which connects the fundamental points $\mathfrak{Q}_i$ of multiplicity x_i so that $\Sigma x_i = n$, corresponds a curve of O^{th} order, which consists merely of a number of separate, singular points associated to the line s as a singular line."

§12.—*The Centre-curve H.*

"The centre-curve H is of the order

$$3\,(n - 1) - \Sigma m_i,$$

and contains any x-ple fundamental point $\mathfrak{Q}$ which is of the multiplicity θ_σ for the singular curves $S_\sigma^{(m)}$, if $x > 1$, as a

$$(3x - \Sigma\theta_\sigma - 2)\text{-ple point,}$$

but a simple fundamental point as a double-point or node, if no singular curve passes through it, otherwise as a point of the multiplicity

$$2 - \Sigma\theta_\sigma."$$

The latter, because the simple fundamental point is a double-point for one curve of the net, while the x-ple fundamental point in general is not a $(x + 1)$-ple point of any curve c of the net K.

"In a simple fundamental point, the centre-curve H has the same two tangents as the one curve of the net which has this point as double-point."

We prove these results by producing the centre-curve H in the same way as in §6, by three projective nets of first polar-curves, but by using as the point a the fundamental point $\mathfrak{O}$ and drawing line g through $\mathfrak{O}$.

"The sum of the orders of the singular curves $S_i^{(m)}$ is

$$\Sigma m_i \leqq 3(n-1)."$$

"If

$$\Sigma m_i = 3(n-1),$$

the correspondence has no centre-curve or self-corresponding curve, but only *separate centre-points* or *self-corresponding points*."

N. B.—In the foregoing we did not specially mention the case that all the curves of the net K had a certain number of *common tangents in a fundamental point*, because this case affords nothing new. For any fundamental point with one or more common tangents can be considered as consisting of, and dissolved into, two or more separate fundamental points lying infinitely near each other.

CHAPTER III.

RESEARCH ON THE EXISTENCE OF UNIVALENT CORRESPONDENCES.

§13.—*Fixed Base-points and Elements of Determination.*

A net of curves of n^{th} order is determined by

$$P = \frac{(n+1)(n+2)}{2} - 3 = \frac{(n-1)(n+4)}{2} \text{ points or elements,}$$

and can contain in maximo

$$Q = n^2 - 2 \text{ fixed points of intersection.}$$

Now one simple fundamental point represents 1 element of determination and 1 point of intersection.

One double fundamental point represents 3 elements of determination and 4 points of intersection.

One triple fundamental point represents 6 elements of determination and 9 points of intersection.

One $\varkappa$-ple fundamental point represents $\frac{\varkappa(\varkappa+1)}{2}$ elements of determination and $\varkappa^2$ points of intersection.

The correspondence is univalent when the number of fixed points of intersection is

$$\Sigma \varkappa_i^2 = Q = n^2 - 2.$$

Then must be either

$$\Sigma \frac{\varkappa_i(\varkappa_i+1)}{2} = P = \frac{(n-1)(n+4)}{2},$$

and the correspondence is fully determined by its fundamental points, or

$$\Sigma \frac{\varkappa_i(\varkappa_i+1)}{2} < P = \frac{(n-1)(n+4)}{2},$$

and the correspondence is not fully determined by its fundamental points.

§14.—*Univalent Correspondences produced by Simple Fundamental Points.*

That a univalent correspondence might be produced by simple fundamental points alone, the condition is

$$\frac{(n-1)(n+4)}{2} \leqq n^2 - 2,$$

hence

$$n \leqq 3:$$

"By simple fundamental points univalent correspondences can be produced only by means of nets of conics and cubics."

$n = 1$: no correspondence defined.

$n = 2$: 2 simple fundamental points. $P = 3$. $\Sigma \frac{\varkappa_i(\varkappa_i+1)}{2} = 2$. The correspondence is not yet defined by the fundamental points alone.

$n = 3$: 7 simple fundamental points. $P = 7$. $\Sigma \frac{\varkappa_i(\varkappa_i+1)}{2} = 7$. The correspondence is just determined by its fundamental points.

§15.—*Univalent Correspondences produced by Multiple Fundamental Points.*

If a fundamental point is of higher multiplicity than $n/2$, only one of this kind of fundamental points can exist, and the other fundamental points must be of a multiplicity equal or lower than $n - \varkappa$.

"It is always possible to determine univalent correspondences by means of nets of curves of n^{th} order with x-ple fundamental points, if $x > n/3$."

For let the correspondence contain δ x-ple points, where $x < n/2$. Then these δ x-ple points represent

$$\delta\frac{x(x+1)}{2} \text{ elements of determination}$$

and $$\delta x^2 \text{ fixed points of intersection.}$$

Now, in neglecting at first the condition that δ is an integer number, and supposing all fundamental points to be x-ple points, the condition of univalence is

$$\delta x^2 = n^2 - 2,$$

but $$\delta\frac{x(x+1)}{2} \leqq \frac{(n-1)(n+4)}{2},$$

whence $$x \geqq \frac{n^2-2}{3n-2}.$$

That is,

"The multiplicity of the fundamental points must be equal to or higher than $\frac{n^2-2}{3n-2}$ to determine a univalent correspondence."

Now, in excluding the case

$$n \leqq 3$$

as considered already, we get

$$\frac{n^2-2}{3n-2} < n/3.$$

Therefore

"$x > n/3$ determines univalent correspondences."*

N. B.—Very different conditions exist in space. There, by a tri-dimensional system of surfaces of n^{th} order, it is impossible to produce univalent involutory correspondences by mere simple or multiple fundamental points (except when $n = 2$), but only the existence of fundamental curves, that is, curves which are common curves of intersection of all the surfaces of n^{th} order of the tri-dimensional system, produces univalent correspondences in space.

---

* This proof is not rigorously correct because we neglected the condition that δ is an integer number. But, by a slight modification, this condition can be accounted for.

CHAPTER IV.

PARTICULAR CORRESPONDENCES.

A.—UNIVALENT CORRESPONDENCES.

§16.—*Perspective Quadratic Correspondence*, $n = 2$.

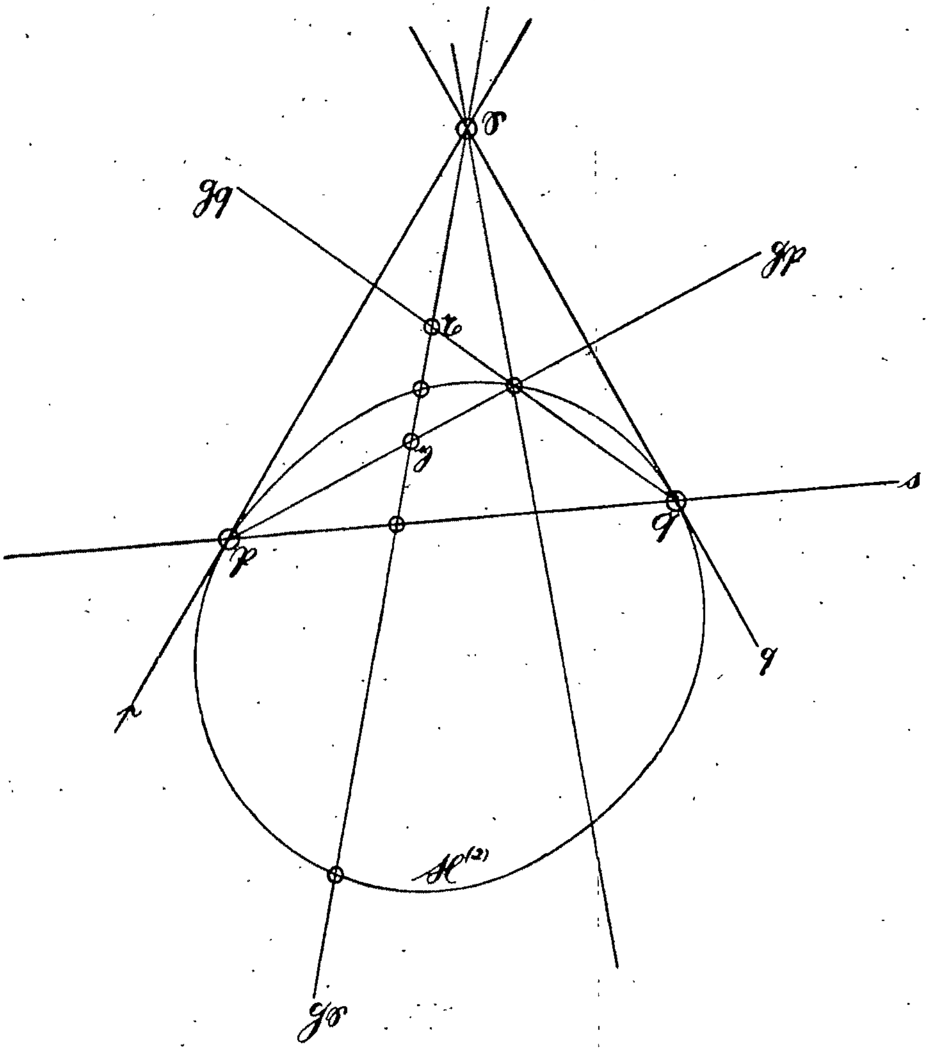


FIG. 4.

A net of conics, which have two points $\mathfrak{P}$ and $\mathfrak{Q}$ as common points of intersection, determines a univalent correspondence, because any point $\mathfrak{x}$ determines a conic pencil which has an additional base-point $\mathfrak{y}$.

This correspondence is not fully determined by its fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$.

The line joining the fundamental points, $s = |\mathfrak{P}\mathfrak{Q}|$, is *singular line* of the correspondence, and therefore corresponds to every line of the plane.

Hence the correspondence is *quadratic*.

With the singular line s is associated as corresponding point the centre $\mathfrak{S}$ of a pencil of rays, which, together with s, form a particular conic pencil of the net. (Fig. 4.)

The *centre-curve* of the net being of the third order, consists of the singular line s and a *conic* $H^{(2)}$, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, and there has as tangents the lines $p = |\mathfrak{S}\mathfrak{P}|$ and $q = |\mathfrak{S}\mathfrak{Q}|$. Hence the singular line s is the polar-line to the singular point $\mathfrak{S}$ with regard to the centre-conic $H^{(2)}$.

To a straight line g corresponds a conic, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$ and through the singular point $\mathfrak{S}$.

To the rays $g_{\mathfrak{P}}$ of the pencil with its centre at the fundamental point $\mathfrak{P}$ correspond the rays $g_{\mathfrak{Q}}$ of the pencil with its centre at the other fundamental point $\mathfrak{Q}$, and inversely. These two corresponding pencils of rays $\mathfrak{P}$ and $\mathfrak{Q}$ are *projective*, and the locus of their points of intersection is the centre-conic $H^{(2)}$.

The rays $g_{\mathfrak{S}}$ of the pencil, with the singular point $\mathfrak{S}$ as the centre, are cut by the corresponding pencils $\mathfrak{P}$ and $\mathfrak{Q}$ in corresponding points $\mathfrak{x}$ and $\mathfrak{y}$, which have the points of intersection of $g_{\mathfrak{S}}$ with $H^{(2)}$ as self-corresponding points, and $\mathfrak{S}$ and $(s, g_{\mathfrak{S}})$ as a pair of corresponding points, and therefore lie in *quadratic involution*.

Thus this quadratic correspondence is *perspective*, with the *singular point* $\mathfrak{S}$ as the *centre of perspectivity*.

The rays of the pencils $\mathfrak{P}$ and $\mathfrak{Q}$ correspond to one another, those of the pencil $\mathfrak{S}$ correspond to themselves and contain quadratic involutions of corresponding points.

To the line s corresponds point $\mathfrak{S}$, and conversely.
" " p " " $\mathfrak{P}$, " "
" " q " " $\mathfrak{Q}$, " "

The corresponding curve of a conic is a unicursal quartic, which has the points $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{S}$ as its double-points.

The corresponding curve of a conic, which passes through one fundamental point $\mathfrak{P}$, is a unicursal cubic, which passes through the points $\mathfrak{P}$ and $\mathfrak{S}$, and has the other fundamental point $\mathfrak{Q}$ as its double-point.

The corresponding curve of a conic, which passes through the singular point $\mathfrak{S}$, is a unicursal cubic, which passes through the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, and has the singular point $\mathfrak{S}$ as its double point.

The corresponding curve of a conic, which passes through both fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$, is another conic which touches the former one in the fundamental points $\mathfrak{P}$ and $\mathfrak{Q}$.

The corresponding curve of a conic, which passes through the singular point $\mathfrak{S}$ and one of the fundamental points $\mathfrak{P}$, is another conic which passes through the other fundamental point $\mathfrak{Q}$ and through the singular point $\mathfrak{S}$.

The corresponding curve of a conic, which passes through the three points $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{S}$, is a straight line, and conversely.

This correspondence is determined by the centre-conic $H^{(2)}$ and the singular point $\mathfrak{S}$. The singular line s is the polar-line of $\mathfrak{S}$ with respect to the centre-conic $H^{(2)}$, and any pair of corresponding points forms an harmonic range with the self-corresponding points $(g_{\mathfrak{S}}, H^{(2)})$, on the rays $g_{\mathfrak{S}}$ of the pencil $\mathfrak{S}$.

§17.—*Correspondence by Reciprocal Radii.*

Of special interest is a particular case of this perspective quadratic correspondence which we derive by assuming as fundamental point $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$ the two *imaginary circular-points at infinity.*

Then all the conics of the net are *circles.*

The *singular line* is the *straight line s_∞ at infinity.*

The *centre-conic* $H^{(2)}$ is a *circle,* because passing through the circular-points $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$ as fundamental points, which has the singular point $\mathfrak{S}$ *as its centre,* because $\mathfrak{S}$ is the pole of the singular line s_∞ at infinity, with respect to the centre-circle $H^{(2)}$.

Let the radius of this centre-circle $= r$.

Any pair of corresponding points $\mathfrak{x}$, $\mathfrak{y}$ forms an harmonic range on a ray $g_{\mathfrak{S}}$, which has the points of intersection of $|\mathfrak{x}\mathfrak{y}| = g_{\mathfrak{S}}$ with the centre-circle $H^{(2)}$ as double-points.

Hence, from the properties of the harmonic range, we derive

$$\mathfrak{S}\mathfrak{x} \times \mathfrak{S}\mathfrak{y} = r^2,$$

the *definition of the correspondence by reciprocal radii.*

To a straight line g corresponds a conic through $\mathfrak{S}$, $\mathfrak{P}^{(i)}$, $\mathfrak{Q}^{(i)}$, that is, a circle through $\mathfrak{S}$ which intersects g upon $H^{(2)}$.

To a circle, that is, a conic through $\mathfrak{P}^{(i)}$ and $\mathfrak{Q}^{(i)}$, corresponds another conic through $\mathfrak{P}^{(i)}$, $\mathfrak{Q}^{(i)}$; that is, a circle.

To a circle through $\mathfrak{S}$ corresponds a straight line, etc.

In this way we derive the properties of the well-known circular correspondence by particularization of the general perspective quadratic correspondence into which it can be transformed by collinear transformation.

§18.—*Correspondence of* 8^{th} *Order*, $n = 3$.

A net of cubics, with 7 common base-points, $\mathfrak{A}$, $\mathfrak{B}$ .... $\mathfrak{G}$, as fundamental points, defines a univalent correspondence which is just determined by its fundamental points.

This correspondence is of the 8th order, and to any straight line corresponds a unicursal curve of 8th order, which has the 7 fundamental points $\mathfrak{A}$, $\mathfrak{B}$.... $\mathfrak{G}$ as triple-points.

The corresponding curve of a straight line, which passes through one of the fundamental points, $\mathfrak{A}$, is a unicursal quintic which passes through $\mathfrak{A}$ as simple point, and has the other 6 fundamental points $\mathfrak{B}$ .... $\mathfrak{G}$ as double-points.

The corresponding curve of one of the 21 connecting lines of two fundamental points, is the conic determined by the other 5 fundamental points.

The centre-curve $H^{(6)}$ of the correspondence is of the 6th order, with the 7 fundamental points $\mathfrak{A}$, $\mathfrak{B}$ .... $\mathfrak{G}$ as double-points.

Here again a particular case of the correspondence is of special interest.

§19.—*The Quadratic Steinerian Correspondence.*

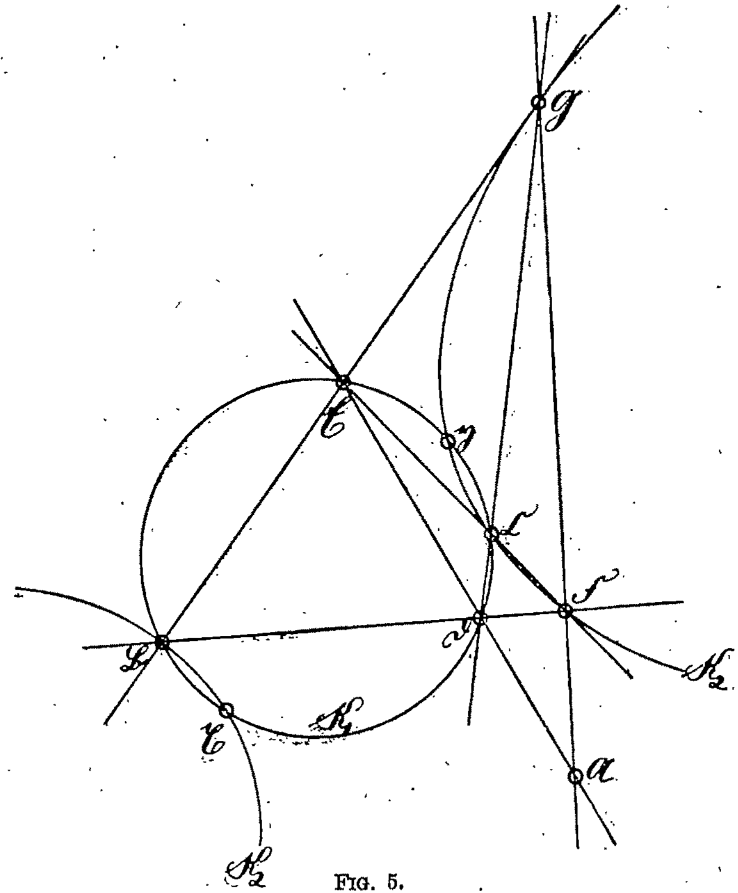


FIG. 5.

Let the 7 fundamental points 𝔄, 𝔅, . . . , 𝔊 of the correspondence considered in §18 lie three upon each of 6 straight lines, so that the configuration of these 7 points forms a *complete quadrangle.*

Let 𝔄, 𝔅, ℭ be the *diagonal points,* 𝔇, 𝔈, 𝔉, 𝔊 the vertices of this quadrangle. Then those 6 lines

|𝔄𝔇𝔈|, |𝔄𝔉𝔊|,
|𝔅𝔇𝔉|, |𝔅𝔊𝔈|,
|ℭ𝔇𝔊|, |ℭ𝔈𝔉|,

disintegrate from all the curves of 8th order of the correspondence as *singular lines.*

Hence the corresponding curve of a straight line is a conic which passes through the diagonal points 𝔄, 𝔅, ℭ, for these points are triple-points in the general $C^{(8)}$, but only double-points in the quadrangle. That is,

"The correspondence is *quadratic.*"

From the centre-curve $H^{(6)}$ these 6 singular lines disintegrate also, and the general centre-curve being of 6th order, we derive

"This quadratic correspondence has *no centre-curve,* but only 4 *separate self-corresponding points,* $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$."

For these 4 points are triple-points in the quadrangle, but only double-points in the general centre-curve.

Every pencil of cubics of this particular net contains 6 special cubics which consist of a singular line and a conic passing through the 4 fundamental points not lying on this singular line.

Any two of those conics have two fundamental points in common.

Hence, to construe the point $\mathfrak{y}$, which corresponds to $\mathfrak{x}$ taken at random in the plane, we make use of these particular cubics of the cubic pencil determined by $\mathfrak{x}$, and thus get the point $\mathfrak{y}$ as fourth point of intersection of two conics, which have the point $\mathfrak{x}$ and two fundamental points in common.

Therefore the construction of $\mathfrak{y}$ is linear.

Thus we get the point $\mathfrak{y}$, which corresponds to $\mathfrak{x}$, as the fourth point of intersection of two conics, which are parts of cubics of the cubic pencil $\mathfrak{x}$, the other part of these special cubics being singular lines.

Let these two special cubics of the cubic pencil $\mathfrak{x}$ be

1). The singular line $|\mathfrak{A}\mathfrak{F}\mathfrak{G}|$ and the conic K_1 passing through $\mathfrak{x}\mathfrak{B}\mathfrak{C}\mathfrak{D}\mathfrak{E}$.
2). " " " $|\mathfrak{A}\mathfrak{D}\mathfrak{E}|$ " " K_2 " " $\mathfrak{x}\mathfrak{B}\mathfrak{C}\mathfrak{F}\mathfrak{G}$.

These two conics K_1 and K_2 intersect in $\mathfrak{x}\mathfrak{B}\mathfrak{C}$ and the corresponding point $\mathfrak{y}$.

Now, by the projective properties of the conic, we get in the conic K_1 the projective pencils of rays, with the centres $\mathfrak{B}$ and $\mathfrak{C}$,

$$\mathfrak{B}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}), \tag{a}$$

in the conic K_2 the projective pencils of rays, with the same centres $\mathfrak{B}$ and $\mathfrak{C}$,

$$\mathfrak{B}\,(\mathfrak{x}\mathfrak{y}\mathfrak{F}\mathfrak{G}) \barwedge \mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{F}\mathfrak{G}), \tag{b}$$

or, what is the same,

$$\mathfrak{B}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{E}\mathfrak{D}); \tag{c}$$

hence (c) combined with (a),

$$\mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) \barwedge \mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{E}\mathfrak{D}),$$

therefore

$$\mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E}) = -1,$$

or $\mathfrak{C}\,(\mathfrak{x}\mathfrak{y}\mathfrak{D}\mathfrak{E})$ forms an harmonic range; that is,

"The pairs of corresponding points $\mathfrak{x}$, $\mathfrak{y}$ of this quadratic involutory correspondence are projected from any diagonal point $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ by the rays of a quadratic involution, which has as its self-conjugates the two singular lines intersecting in the diagonal point."

Now the construction of the corresponding point $\mathfrak{y}$ is the following: Connect $\mathfrak{x}$ with two diagonal points $\mathfrak{B}$ and $\mathfrak{C}$ and produce the fourth harmonic rays with respect to the pairs of singular lines intersecting in $\mathfrak{B}$ and $\mathfrak{C}$ to $|\mathfrak{B}\mathfrak{x}|$ and $|\mathfrak{C}\mathfrak{x}|$. These fourth harmonic rays intersect in $\mathfrak{y}$, which corresponds to $\mathfrak{x}$ in this quadratic correspondence.

But this is the construction of the *Steinerian quadratic involutory correspondence*, which has the diagonal fundamental points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ as its base-points, and the four other fundamental points $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$ as its self-corresponding points, or as the base-points of that conic pencil, the conjugate poles of which are the corresponding points of the Steinerian correspondence, so that we need not enter further into the consideration of this particular correspondence.

Therefore, we get a new property of the Steinerian correspondence—

"Any pair of corresponding points of a Steinerian correspondence gives with the three base-points and the four self-corresponding points of the correspondence, a set of 9 base-points of a cubic pencil."

§20.—*Correspondence of 5th Order*, $n = 3$.

Another univalent correspondence is defined by a net of cubics with one double fundamental point $\mathfrak{D}$ and three simple fundamental points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, but is not fully determined by its fundamental points.

This correspondence is of the 5th order, and contains three singular lines, $|\mathfrak{D}\mathfrak{A}|$, $|\mathfrak{D}\mathfrak{B}|$, $|\mathfrak{D}\mathfrak{C}|$, every one of which is associated with one singular point, $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$.

The centre-curve is a cubic $H^{(3)}$, passing through $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ as simple points.

The corresponding curve of a straight line is a unicursal quintic, which has the point $\mathfrak{D}$ as triple-point, the points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ as double-points, and passes through the singular points $\mathfrak{A}_1$, $\mathfrak{B}_1$, $\mathfrak{C}_1$ as simple points.

§21.—*Nets of Quartics.*

Number of fixed points of intersection, $Q = n^2 - 2 = 14$.

Number of elements of determination, $P = \frac{(n-1)(n+4)}{2} = 12$.

The quartic net gives the following univalent correspondences:

1). 2 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 simple fundamental points, $\mathfrak{P}_i$.

The correspondence is fully determined thereby.

14th Order.—Centre-curve, $H^{(8)}$ of 8th order, with 2 triple-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 double-points, $\mathfrak{P}_i$.

One singular line, $s = |\mathfrak{D}_1\mathfrak{D}_2|$, with one associated singular point, $\mathfrak{S}$, which, together with $\mathfrak{D}_1\mathfrak{D}_2$ and $\mathfrak{P}_i$, gives a set of 9 base-points of a cubic pencil.

The general curve $C^{(14)}$ has two points, $\mathfrak{D}_1$, $\mathfrak{D}_2$ of the multiplicity 7; 6 quadruple points $\mathfrak{P}_i$; and passes through $\mathfrak{S}$.

2). 3 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, and two simple fundamental points, $\mathfrak{P}_1$, $\mathfrak{P}_2$.

The correspondence is not fully determined thereby.

8th Order.—Centre-curve, $H^{(4)}$ of 4th order, with $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{P}_1$, $\mathfrak{P}_2$ as simple points.

3 singular lines, $|\mathfrak{D}_1\mathfrak{D}_2|$, $|\mathfrak{D}_1\mathfrak{D}_3|$, $|\mathfrak{D}_2\mathfrak{D}_3|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and

1 singular conic through the 5 fundamental points, with one associated point $\mathfrak{S}^{(2)}$.

The general curve $C^{(8)}$ has 3 quadruple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, 3 double-points, $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{S}^{(2)}$, and passes through the 3 singular points, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$.

3). 1 triple-point $\mathfrak{T}$ and 5 simple fundamental points $\mathfrak{P}_i$.

The correspondence is not fully determined thereby.

10th Order.—Centre-curve $H^{(4)}$ of 4th order, with $\mathfrak{T}$ as double-point and $\mathfrak{P}_i$ as simple points.

5 singular lines $s_i = |\mathfrak{T}\mathfrak{P}_i|$, each with one associated singular point $\mathfrak{S}_i$.

The general curve $C^{(10)}$ has a point of the multiplicity 7 in $\mathfrak{T}$, 5 triple points in $\mathfrak{P}_i$, and passes through $\mathfrak{S}_i$ as simple points.

§22.—*Nets of Quintics.*

Number of fixed points of intersection, $Q = n^2 - 2 = 23$.

Number of elements of determination, $P = \dfrac{(n-1)(n+4)}{2} = 18$.

The quintic net gives the following univalent correspondences:

1). 5 double-points, $\mathfrak{D}_i$, and 3 simple points, $\mathfrak{P}_i$; fully determined.

20th Order.—Centre-curve, $H^{(10)}$ of 10th order, with 5 triple-points, $\mathfrak{D}_i$, and 3 double-points, $\mathfrak{P}_i$.

One singular conic through the 5 double-points, $\mathfrak{D}_i$, with one associated singular point, $\mathfrak{S}^{(2)}$.

General curve, $C^{(20)}$, with 5 8-ple points, $\mathfrak{D}_i$, 3 quintuple points, $\mathfrak{P}_i$, and one double-point, $\mathfrak{S}^{(2)}$.

2). 1 triple-point, $\mathfrak{T}$, 2 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 simple points, $\mathfrak{P}_i$; fully determined.

22d Order.—Centre-curve, $H^{(10)}$ of 10th order, with $\mathfrak{T}$ as quintuple point, 2 triple-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, and 6 double-points, $\mathfrak{P}_i$.

2 singular lines, $|\mathfrak{T}\mathfrak{D}_1|$ and $|\mathfrak{T}\mathfrak{D}_2|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$.

General curve, $C^{(32)}$, with 13-ple point, $\mathfrak{T}$, 2 9-ple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, 6 quintuple points, $\mathfrak{P}_i$, and 2 simple points, $\mathfrak{S}_1$, $\mathfrak{S}_2$.

3). 1 triple-point, $\mathfrak{T}$, 3 double-points, $\mathfrak{D}_1$, $\mathfrak{D}_2$; $\mathfrak{D}_3$, and 2 simple points, $\mathfrak{P}_1$, $\mathfrak{P}_2$; undetermined.

13th Order.—Centre-curve, $H^{(5)}$ of 5th order, with double-point, $\mathfrak{T}$, and simple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{P}_1$, $\mathfrak{P}_2$.

3 singular lines, $|\mathfrak{T}\mathfrak{D}_1|$, $|\mathfrak{T}\mathfrak{D}_2|$, $|\mathfrak{T}\mathfrak{D}_3|$, each with one associated singular point, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and

2 singular conics, each passing through $\mathfrak{T}$, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$ and one of the simple points, $\mathfrak{P}_i$, each with 2 associated singular points, $\mathfrak{S}_1^{(2)}$, $\mathfrak{S}_2^{(2)}$; $\mathfrak{S}_{11}^{(2)}$, $\mathfrak{S}_{21}^{(2)}$.

General curve, $C^{(13)}$, with 8-ple point, $\mathfrak{T}$, 3 quintuple points, $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, 2 triple-points, $\mathfrak{P}_1$, $\mathfrak{P}_2$, 4 double-points, $\mathfrak{S}_1^{(2)}$, $\mathfrak{S}_2^{(2)}$, $\mathfrak{S}_{11}^{(2)}$, $\mathfrak{S}_{21}^{(2)}$, and 3 simple points, $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$.

4). 1 quadruple point, $\mathfrak{Q}$, and 7 simple points, $\mathfrak{P}_i$; undetermined.

17th Order.—Centre-curve, $H^{(5)}$ of 5th order, with triple-point, $\mathfrak{Q}$, and 7 simple points, $\mathfrak{P}_i$.

7 singular lines, $|\mathfrak{Q}\mathfrak{P}_i|$, each with one associated point, $\mathfrak{S}_i$.

General curve, $C^{(17)}$, with 13-ple point, $\mathfrak{Q}$, 7 quadruple points, $\mathfrak{P}_i$, and 7 simple points, $\mathfrak{S}_i$.

§23.—*Nets of Sextics.*

$Q = 34. \qquad P = 25.$

1).	1 triple	point,	6	double points,	1	simple point;	fully determined.	35th	order.
2).	2 "	"	3	"	4	"	" "	30th	"
3).	2 "	"	4	"	–	"	undetermined.	18th	"
4).	3 "	"	–	"	7	"	fully determined.	32d	"
5.)	3 "	"	1	"	3	"	undetermined.	20th	"
6).	1 quadruple "		3	"	6	"	fully determined.	32d	"
7).	1 "	"	3	"	2	"	undetermined.	27th	"
8).	1 quintuple "		–	"	9	"	"	26th	"
etc.							etc.		

B.—MULTIVALENT CORRESPONDENCES.

§24.—*Trivalent Cubic Correspondence*, $n = 2$.

A net of conics, which have no common point of intersection, determines a trivalent cubic correspondence, which has a cubic as its centre-curve, $H^{(3)}$.

To any point $\mathfrak{x}$ of the plane correspond 3 points $\mathfrak{y}_1$, $\mathfrak{y}_2$, $\mathfrak{y}_3$, which, when $\mathfrak{x}$

moves along a straight line g, produce a cubic curve $C^{(3)}$. This cubic $C^{(3)}$ intersects the straight line g in 3 self-corresponding points.

In general, a line g contains no pairs of conjugate or mutually corresponding points, But if a line g_0 contains one pair of points which correspond to one another, the line g_0 contains an infinite number of pairs of mutually corresponding points, which produce a *quadratic involution* on the line g_0.

Such a line may be called a *self-corresponding line.*

To any one of the self-corresponding lines g_0 corresponds (besides g_0) a conic; and all the self-corresponding lines g_0 envelop a curve of the third class.

For, the curve $C^{(3)}$, corresponding to a straight line g, is produced as the locus of points of intersection of two conic pencils of the net, which have one conic in common, and are put into correspondence by the points of the line g, so that to a conic of the one pencil always correspond two conics of the other pencil. These two conic pencils cut the line g in two quadratic involutions, which have as their common pair of elements the two points of intersection of g with the conic, which is common to both conic pencils.

If the line g contains a pair of mutually corresponding points, this is a common pair of elements of the two quadratic involutions also, and these involutions, therefore, are identically the same; that is, all their pairs of elements are mutually corresponding points on g, and g is a self-corresponding ray.

All the self-corresponding rays passing through a point $\mathfrak{x}$ contain a point corresponding to $\mathfrak{x}$ in their quadratic involution, and therefore in the cubic correspondence. Thus they are the three lines joining $\mathfrak{x}$ with its corresponding points $\mathfrak{y}_1$, $\mathfrak{y}_2$, $\mathfrak{y}_3$. Hence the self-corresponding rays envelop a curve of third class, q. e. d.

§25.—*Bivalent Correspondence of 4^{th} Order, $n = 3$.*

6 points, $\mathfrak{P}_1, \mathfrak{P}_2 \dots \mathfrak{P}_6$ on a conic K determine as simple fundamental points of a cubic net a bivalent correspondence which has the conic K as singular curve, and therefore is of 4th order.

With the singular conic K is associated a singular point $\mathfrak{S}^{(2)}$.

This correspondence is *perspective*, with the singular point $\mathfrak{S}^{(2)}$ as the *centre of perspectivity*. The two points $\mathfrak{y}_1$, $\mathfrak{y}_2$, corresponding to $\mathfrak{x}$, lie on the straight line connecting $\mathfrak{x}$ with $\mathfrak{S}^{(2)}$. These lines of the pencil $\mathfrak{S}^{(2)}$ are *self-corresponding lines*, and contain *cubic involutions* of conjugate or mutually corresponding points.

The centre-curve $H^{(4)}$ is a quartic, passing through the 6 fundamental points.

The general curve $C^{(4)}$ passes through the 6 fundamental points, and has the singular point $\mathfrak{S}^{(2)}$ as a double-point.

CONTENTS.

INTRODUCTION:

§1. Ranges of Points, ... 39
§2. Planes in Correspondence, ... 42

CHAPTER I.—CORRESPONDENCE DEFINED BY A GENERAL NET.

§3. Definition and Quantivalence of the Correspondence, ... 43
§4. The Order of the Correspondence, ... 44
§5. Points and Lines in the Correspondence, ... 45
§6. The Centre-curve $H^{3(n-1)}$ of the Correspondence, ... 46

CHAPTER II.—CORRESPONDENCE DEFINED BY A NET WITH FIXED BASE-POINTS OR FUNDAMENTAL POINTS.

§7. Definition and Quantivalence, ... 47
§8. Singular Curves and Singular Points, ... 48
§9. The Order of the Correspondence, ... 49
§10. The General Curve C, ... 50
§11. Particular Curves C, ... 51
§12. The Centre-curve H, ... 52

CHAPTER III.—RESEARCH ON THE EXISTENCE OF UNIVALENT CORRESPONDENCES.

§13. Fixed Base-points and Elements of Determination, ... 53
§14. Univalent Correspondences produced by Simple Fundamental Points, ... 54
§15. Univalent Correspondences produced by Multiple Fundamental Points, ... 54

CHAPTER IV.—PARTICULAR CORRESPONDENCES.

A: UNIVALENT CORRESPONDENCES.

§16. Perspective Quadratic Correspondence, $n=2$, ... 56
§17. Correspondence by Reciprocal Radii, ... 58
§18. Correspondence of 8th order, $n=3$, ... 59
§19. The Quadratic Steinerian Correspondence, ... 60
§20. Correspondence of 5th order, $n=3$, ... 62
§21. Nets of Quartics, ... 62
§22. Nets of Quintics, ... 63
§23. Nets of Sextics, ... 64

B: MULTIVALENT CORRESPONDENCES.

§24. Trivalent Cubic Correspondence, $n=2$, ... 64
§25. Bivalent Correspondence of 4th order, $n=3$, ... 65

On the Algebraic Proof of a Certain Series.[*]

By Emory McClintock.

The following series was published in 1879 (*American Journal of Mathematics*, II, 108):

$$\log x = y + \frac{2a-1}{2} y^2 + \frac{2a-1}{2} \frac{2a-2}{3} y^3 + \dots . \tag{1}$$

Here $y = x^{1-a} - x^{-a}$. In presenting this series at that time I stated that it might be obtained by the aid of Lagrange's theorem, or by the aid of another expansion-theorem, somewhat simpler than that of Lagrange though less comprehensive, which I published (p. 147) in the same paper; and I added that the "temporary lack" of an algebraic demonstration was "certainly to be regretted." I am now able to supply the desired algebraic proof.

It was shown in the paper referred to that, writing x^h for x, (1) is equivalent to

$$\log x = h^{-1}\left(y + \frac{2a-1}{2} y^2 + \dots\right), \tag{2}$$

where $y = x^{h-ha} - x^{-ha}$; that when $a = 0$ and $a = 1$ respectively we have as special cases of (1) the known series

$$\log x = (x-1) \quad - \frac{1}{2}(x-1)^2 \quad + \frac{1}{3}(x-1)^3 \quad - \dots , \tag{3}$$

$$\log x = (1-x^{-1}) + \frac{1}{2}(1-x^{-1})^2 + \frac{1}{3}(1-x^{-1})^3 + \dots ; \tag{4}$$

that when $a = m/n$, a proper fraction, the coefficients of y^n, y^{2n}, etc., disappear; that, in particular, when $a = \frac{1}{2}$, the even powers of y disappear, whence, writing $2t$ for y, so that $t = \frac{1}{2}(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$,

---

[*] An abstract of this paper was read before the New York Mathematical Society on March 6, 1891.

$$\log x = 2\left[t - \frac{1}{2}\,\frac{t^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\,\frac{t^3}{5} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,\frac{t^5}{7} + \ldots\right], \tag{5}$$

a series possibly novel for real values of x, though known when x is a certain symbol of operation (the usual series for $z = \text{arc}\sin u$ may be obtained from [5] as a special case if $x = e^{2iu}$); that (5) is a remarkably convergent logarithmic series; that if, in (2), h tends towards 0, we have

$$\log x = \lim\,(x^{h-ha} - x^{-ha})/h, \tag{6}$$

a general limit-expression of which $\log x = \lim\,(x^h - 1)/h$ is a known case; and that if $a = -h^{-1}$, and h tends towards 0, $\lim y/h = -\lim ay = \lim\,(x^{h+1} - x)/h = x\lim\,(x^h - 1)/h = x\log x$, so that (2) becomes

$$\log x = v - v^2 + 3v^3/2! - 4^2v^4/3! + 5^3v^5/4! - \ldots, \tag{7}$$

where $v = x\log x$.

As regards the validity of these results, the general series (1) will be found convergent when (using mod to express mere arithmetical value)

$$\text{mod}\,(x-1)\,x^{-a} < \text{mod}\,(a-1)^{a-1}a^{-a};$$

and (2) will be found convergent when

$$\text{mod}\,y < \text{mod}\,(a-1)^{a-1}a^{-a},\ \text{ where }\ y = x^{h-ha} - x^{-ha}.$$

Applying this statement to special cases, we may say that (1) is convergent if $a \geqq 1$ when $x > 1$, or if $a \leqq 0$ when $0 < x < 1$, unless perhaps when $x - a/(a-1)$, in which case $(x-1)\,x^{-a} = (a-1)^{a-1}a^{-a}$; also, that (1) is convergent if $a = \frac{1}{2}$ when $\text{mod}\,y < 2$, that is, when x lies between the two values of $3 \pm 2\sqrt{2}$. In the case $(a = -\infty)$ represented in (7), where (2) is modified by putting $a = -h^{-1}$ and diminishing h towards 0, the application of this criterion shows (7) convergent when $\text{mod}\,v < e^{-1}$, because $v = \lim y/h$, and we must have

$$\text{mod}\,y/h < \text{mod}\,(a-1)^{a-1}a^{-a}/h,\ \text{ where } a = -h^{-1},$$

the limit of the second member being e^{-1}. It is easy to prove convergency in any of these special cases, or in any case in which a is an integer; and I have assumed the truth of the criterion for all other cases. For elementary presentation, however, it is enough to say that (1) is convergent when y is sufficiently small, that is, when the value of x has a certain range according to the given value of a.

The two known series, (3) and (4), are, one or the other, valid for all values of x. The algebraic proof of (1) which I have now to present consists in showing, first, its equivalence to (3) whenever both (1) and (3) are interpretable,* and, secondly, its equivalence to (4) whenever both (1) and (4) are interpretable. If a is assigned, any value of x which makes (1) interpretable will make either (3) or (4) interpretable, so that by this means we have a demonstration of (1) for all interpretable cases.

Let us suppose that (excluding $x = 2$) x and a are such that both (1) and (3) are interpretable. For any such value of x the binomial series for $(1+z)^{-a}$, where $z = x - 1$, is known to be valid; for, if z is a quantity, mod $z < 1$, and if z is an operation affecting a function of t, say $\psi(t)$, it and $\psi(t)$ must be such that $z^{n+1}\psi(t)$ is arithmetically less than $z^n\psi(t)$, the binomial series being interpretable in both cases. Employing that series, therefore, after observing that $y = x^{1-a} - x^{-a} = z(1+z)^{-a}$, and denoting the second member of (1) by $f(y)$, we have

$$\left.\begin{aligned} f(y) &= y + \frac{2a-1}{2}y^2 + \frac{3a-1}{2}\,\frac{3a-2}{3}y^3 + \dots \\ &= z\Big(1 - az + \frac{1}{1.2}[a+1]^{2)}z^2 - \dots\Big) \\ &\quad + \frac{1}{1.2}(2a-1)z^2\Big(1 - 2az + \frac{1}{1.2}[2a+1]^{2)}z^2 - \dots\Big) \\ &\quad + \frac{1}{1.2.3}(3a-1)^{2)}z^3(1 - 3az + \dots) \\ &\quad + \dots\dots\dots\dots\dots\dots \end{aligned}\right\} \qquad (8)$$

Here $(a+1)^{2)}$ means $(a+1)a$, and in general $c^{n)}$ means

$$c(c-1)(c-2)\dots(c-n+1).$$

---

* A series of quantities is interpretable when it is convergent. A series of symbols of operation is interpretable when, if the operations indicated are performed upon a given function, the result is a convergent series. Thus, the exponential series $1 + x + x^2/2! + \dots$ has always a meaning when x is a quantity, but not always when x is the symbol of differentiation (Taylor's theorem). For all series *ad infinitum* (i. e. without remainder) a convention is necessary that the symbols can have no values assigned so as to render such series uninterpretable. Subject to this restriction, any power-series convergent for small values of the variable may be handled freely as long as the variable has no meaning assigned to it.

Collecting the several terms composing the coefficient of z^m, we find them to be identical with the several terms of $\phi(m, ma-1)/m! - (-1)^m/m$, where $\phi(m, v)$ means $v^{m-1)} - m(v-k)^{m-1)} + \frac{1}{1.2}m^{2)}(v-2k)^{m-1)} - \ldots$, in which $k = a-1$. I shall prove that, for all positive integral values of m, and for all values of v and k, $\phi(m, v) = 0$. Assuming this for the moment, it follows that the coefficient of z^m in (8) is $(-1)^{m-1}/m$, whence

$$f(y) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots.$$

The second member of this is the second member of the known equation (3); hence, for all values of y which make both (1) and (3) interpretable, $f(y) = \log x$, and (1) is true.

Again, let us suppose that x and a are such that both (1) and (4) are interpretable, in which case the binomial series for $(1-u)^{a-1}$, where $u = 1 - x^{-1}$, is also interpretable. Since $y = x^{1-a} - x^{-a} = u(1-u)^{a-1}$, we shall find, on substituting this for y in $f(y)$, and performing the binomial expansions indicated, that the coefficient of u^m is identical, term by term, with $\phi(m, ma-1) + 1/m$, wherein $k = a$. Since $\phi(m, v) = 0$, the coefficient of u^m is $1/m$, so that

$$f(y) = u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \ldots.$$

The second member of this is the second member of the known equation (4); hence, for all values of y which make both (1) and (4) interpretable, $f(y) = \log x$, and (1) is again true.

The necessary algebraic proof of the known theorem in finite differences that $\phi(m, v) = 0$ may be supplied as follows. By definition,

$$\phi(m, v) = v^{m-1)} - m(v-k)^{m-1)} + m\frac{m-1}{2}(v-2k)^{m-1)} - \ldots.$$

Let the multiplications indicated in $(v+k-m+1)\phi(m, v+k)$ and in $(v-mk-m+1)\phi(m, v)$ be performed, and the first term of the latter series placed beneath the second term of the former, and so on, thus:

$$(v+k)^{m)} - m(v+k-m+1)v^{m-1)} + m\frac{m-1}{2}(v+k-m+1)(v-k)^{m-1)}$$
$$- \ldots . (v-mk-m+1)v^{m-1)} - m(v-mk-m+1)(v-k)^{m-1)} + \ldots.$$

Subtracting the lower series from the upper, term by term, after due consideration of character of the respective progressions, we find the difference to be identical with $\phi(m+1, v+k)$. In brief,

$$\phi(m+1, v+k) = (v+k-m+1)\,\phi(m, v+k) - (v-mk-m+1)\,\phi(m, v). \quad (9)$$

If, then, we know for any given positive integral value of m, and for all values of v and k, that $\phi(m, v) = 0$, we learn by (9) that the same is true for the value of m next higher, and therefore successively for all higher integral values of m. But we may see at once that $\phi(2, v) = 0$, as well as $\phi(1, v) = 0$, for all values of v and k, since $\phi(1, v) = 1 - 1$, and $\phi(2, v) = v - 2(v - k) + (v - 2k)$; therefore $\phi(m, v) = 0$ for all positive integral values of m.

On Independent Definitions of the Functions log (x) and e^x.[*]

BY EMORY McCLINTOCK.

Twelve years ago there appeared in the second volume of the *American Journal of Mathematics* "An Essay on the Calculus of Enlargement," in which I presented, and urged the acceptance of, a certain unified view of several branches of mathematical science. Giving the name of Enlargement to that operation (E^h) by which $\phi(x)$ becomes $\phi(x+h)$, corresponding to the symbolic equation $E^h\phi(x) = \phi(x+h)$, I remarked that the symbolic algebra of the rational functions of E, commonly known as the Calculus of Finite Differences, and the symbolic algebra of the logarithmic functions of E, commonly known as the Differential Calculus, were really parts of one symbolic algebra of the functions of E, for which I suggested the name of Calculus of Enlargement. After observing that the connecting link between the theory of differentiation and the theory of finite differences had long been thought to be furnished by the equation $E = e^D$ where D means $\frac{d}{dx}$, the symbol of differentiation, I argued that the converse view, $D = \log(E)$, would be preferable because "of the two operations, the simpler should be defined the earlier." Plainly, $E^h\phi(x) = \phi(x+h)$ is a simpler statement than $D\phi(x) = \lim\,[\phi(x+h) - \phi(x)]/h$ when h is indefinitely reduced. "These operations, E and D, are functions of each other, and whichever is defined last must be expressed in terms of the other." "The Calculus of Enlargement regards E as the fundamental symbol, and takes cognizance of other symbols only in case they are, and because they are, functions of E." "The algebra of the functions of E is subject to all the laws of ordinary algebra; and the theory of differentiation is that part of the calculus which corresponds to the theory of logarithms in algebra."

---

[*] An abstract of this paper was read before the New York Mathematical Society on March 6, 1891.

That the acknowledged correspondence between symbolic algebra and ordinary algebra might be brought out in the strongest light, it was then urged that the customary mode of presenting the theory of logarithms be so changed as to make logarithms, as such, more intelligible. The prevailing obscurity was illustrated by quoting De Morgan's sweeping statement that "the only definition of $\log(x)$ used in analysis is y, where $e^y = x$." At first sight this definition is not satisfactory. It is true that, by convention, e^y means a certain limit, or a certain series, and not, except when y is a real quantity, a power of the constant e. While acknowledging the correctness of such customary indirect definitions of $\log(x)$, I ventured to propose concurrently other possible definitions, and among them the known equation $\log(x) = \lim (x^h - 1)/h$, which has since, I am glad to see, been mentioned as a feasible definition by Mr. Glaisher in the article "Logarithms" in the Encyclopædia Britannica. The definitions which were then suggested concurrently for $\log(x)$ all tended to throw light on the nature of the logarithm, and were all, of course, susceptible of subsequent identification. The discussion of logarithms as such was not, however, essential to the chief object then in hand, and for that reason, perhaps, I failed to carry out at that time the notion of concurrent definition to its logical consequences.

In truth, what we may call "the method of concurrent definition" has probably not hitherto been formulated as a scientific method of procedure. The long-standing idea of a definition is that it indicates the thing defined, and that from it, as an unchangeable basis, all other properties must be deduced. We are at liberty to begin with any given relation of a function as a definition, but having once chosen it, we are to adhere to it, since the very essence of a definition appears to be that it at least is definite and unchangeable, so that two simultaneous definitions of one idea would seem logically monstrous. To put the matter in form, then, let us say that "the method of concurrent definition" comprises the definition of $f_1(x)$, with an illustration of its nature, and the separate and independent definition of $f_2(x)$ in like manner, followed by proof that $f_1(x)$ and $f_2(x)$ are identical. Intrinsically, no process can be more logical; and when we come to reflect upon it, we shall find that we have, upon occasion, been practising it all our lives.

Take for instance the best known of all functions, the binomial function $f_1(x, y, m) = (x + y)^m$. Starting from this definition, Newton and his followers for a century undertook to deduce from it the binomial series, with more or less success, until Euler brought out a better statement of the case, employing in substance the principle of concurrent definition. Denoting the series, say, as $f_3(x, y, m)$, he took it up as an independent function, proved certain properties, and then identified f_3 with f_1.

In another paper (*ante*, p.), I have supplied an algebraic proof of a series equivalent to $\log(x)$, for which series, when first presented as an expansion, I could give no better demonstration than that afforded by Lagrange's theorem. The new proof begins with the consideration of the series as a separate function and ends with its identification with known equivalents of $\log(x)$.

It used to be customary to expand e^x, or $\lim(1+h)^{x/h}$ when h tends towards 0, by means of the binomial series in order to obtain the exponential series $1+x+x^2/2!+\ldots$. A better way was devised by Cauchy, beginning with the latter series as a separate subject for examination, deducing its properties, and finally identifying it with e^x. The series is now, indeed, adopted by some of the highest authorities as the original definition of the symbol e^x, sometimes written $\exp(x)$; although others adhere to the limit as the proper definition. Thus in the article "Function" in the Encyclopædia Britannica the series is employed as the definition by Cayley, while in the article "Trigonometry" the limit is employed by Hobson. There is no occasion for controversy. The limit is a definable function possessed of certain properties which may be discussed; the series is another, and the two may be readily identified. Yet I have not met with any distinct announcement of the utility of assigning coordinate rank to these two methods of presenting e^x; those who employ the series having apparently no interest in the limit, and those who begin with the limit appearing to regard the series as a subsidiary deduced expression, for the acquisition of which Cauchy's method is, as he meant it to be, merely an incidental device.

To illustrate more clearly the two views now prevalent, the following summaries of the definitions contained in the Encyclopædia articles just mentioned will be found interesting. The article "Trigonometry" defines e^x as $\lim(1+x/m)^m$, where m tends towards infinity, and $e^{x+\iota y}$ as $\lim(1+[x+\iota y]/m)^m$, so that, putting

$$1+x/m=r\cos\theta,\ \text{and}\ y/m=r\sin\theta,\ e^{x+\iota y}=\lim r^m(\cos m\theta+\iota\sin m\theta),$$

by De Moivre's theorem. Since

$$\begin{aligned} r^2=r^2\cos^2\theta+r^2\sin^2\theta,\ \lim r^m &= \lim(1+2x/m+x^2/m^2+y^2/m^2)^{m/2} \\ &= \lim(1+2x/m)^{m/2}=e^x. \end{aligned}$$

Also, $\lim m\theta = \lim m \operatorname{arc}\tan y/(x+m) = \lim my/(x+m) = y.$

Hence $e^{x+\iota y} = e^x(\cos y + \iota \sin y)$. The expansion of e^x is assumed known, by algebra. On the other hand, the article "Function" defines $\exp(x)$ to be $1 + x + x^2/2! + \ldots .$, where x need not be real, $\cos(x)$ to be $1 - x^2/2! + \ldots .$, and $\sin(x)$ to be $x - x^3/3! + \ldots .$, and deduces the theory of exponential, circular, hyperbolic, and logarithmic functions from these definitions without mention of the limit-expression.

If we attempt to embrace both views at once by declaring both definitions useful, we shall have on the one hand $f_1(x) = \lim(1 + hx)^{1/h}$, as h tends towards 0, and on the other hand $f_2(x) = 1 + x + x^2/2! + \ldots .$ It is easy to prove that $[f_1(1)]^x = f_1(x)$, and that $[f_2(1)]^x = f_2(x)$, so that $f_1(x)$ and $f_2(x)$ are identical if $f_1(1) = f_2(1)$. Let $f_1(1) = f_2(c)$. Let each of these be raised to the power h, let 1 be subtracted, and let the respective remainders be divided by h, after which let h be reduced indefinitely in value; the result in the one case is 1, in the other c, for $[f_2(c)]^h = f_2(ch)$, so that $c = 1$, and the functions are identical. We shall find thus that the connecting link between these two definitions of e^x is y, where $x = \lim(y^h - 1)/h$. Hitherto, I think, the two expressions have been connected, whenever necessary, by employment of the binomial series.

Of the six equations following, the third, fourth, and fifth represent known definitions, and the others represent some of the known identities which, in the earlier paper, I suggested for use in the future as definitions:

$$\phi_1(x) = \log(x) = \lim(x^h - 1)/h, \qquad (1)$$

$$\phi_2(x) = \log(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \ldots ., \qquad (2)$$

$$\log(x) = y, \text{ where } e^y = x, \qquad (3)$$

$$f_1(y) = \exp(y) = e^y = \lim(1 + yh)^{1/h} = \lim(1 + k)^{y/k}, \qquad (4)$$

$$f_2(y) = \exp(y) = e^y = 1 + y + y^2/2! + \ldots ., \qquad (5)$$

$$\exp(y) = e^y = x, \text{ where } y = \log x. \qquad (6)$$

It is known that the exponential property, $e^u e^v = e^{u+v}$, is a direct result of the definition (4), and that it may be derived, as by Cauchy, from (5), by multiplication of the two series. I shall show similarly that the logarithmic property, $\log(x^n) = n\log(x)$, is a direct result of (1), and that it may also be derived from (2). I have already noted that (1) is the natural connecting link between (4) and (5); and I shall show similarly that (4) is the natural connecting link between (1) and (2).

There is indeed no good reason for the priority hitherto given to e^y in relation to $\log(x)$. It is probably still true that "the only definition of $\log(x)$ used in analysis is y, where $e^y = x$," but we are at liberty to change the custom if it is not a good one. The first notion of logarithms was due to the observed fact that powers form a geometrical series while the numbers denoting the powers form an arithmetical series; and the first practical notion that any child obtains is that the "common logarithm" is the index of that power of 10 which is equal to the "number corresponding." Our early familiarity with the equation $10^{\lambda(n)} = n$, where λ represents the "common logarithm," is a sufficient explanation of the historical fact that, by custom, $\log(x)$ means "y, where $e^y = x$." The origin of the custom is natural and obvious, but the question is whether it is a necessary custom. What is e, and what is y? The analytic logarithm defined as "y, where $e^y = x$," is an incommensurable power, whatever that may be—for all the powers at first known to us are commensurable—of an incommensurable quantity. Is this really the simplest explanation we can get of $\log(x)$, even when x is real and positive? But the use of the functional symbols e^y, $\log(x)$, has gone far beyond that restriction. We have seen how e^y has to be defined when y is complex; the idea of powers, at first so natural, so apparently simple, is flatly abandoned. We shall now see that for all quantities, real or complex, $\log(x)$ is a function at least as easily apprehended as e^y; while it is known—and this, as the reader will have understood, is the strongest motive with the present writer—that in symbolic algebra E is simpler than D, and therefore $D = \log(E)$ should have priority, as an analytical statement, over $E = e^D$. For the presentation of the true definition of differentiation, $D = \log(E)$, we need to have some better algebraic definition of the logarithm than "y, where $e^y = x$." There is a recognized mechanical incongruity when the cart precedes the horse.

Referring to (1), let us adopt as a definition

$$\log(x) = \lim (x^h - 1)/h, \tag{1}$$

where h tends towards 0. Let $x_1 = x^h - 1$, and $u_1 = u^h - 1$, so that $\lim x_1 = 0$. Then

$$\begin{aligned}\log(xu) &= \lim (x^h u^h - 1)/h = \lim ([1 + x_1][1 + u_1] - 1)/h \\ &= \lim x_1/h + \lim u_1/h + \lim x_1 . \lim u_1/h \\ &= \log(x) + \log(u).\end{aligned} \tag{7}$$

This fundamental property of logarithms is thus shown to be obtained at once from (1) as a definition. It will be proved (Appendix A) that from (7) we have readily

$$\log(x^n) = n\log(x), \tag{8}$$

where n is any commensurable number, and where x is any symbol subject to algebraic laws.

If e is the number whose logarithm is 1, we find from (8) that

$$\log(e^n) = n. \tag{9}$$

This is equivalent to (6) as an inverse definition of e^n. But, more directly from (1), if $y = \log x$, we have $y = \lim x_1/h$, and since $x^h = 1 + x_1$,

$$x = (1 + x_1)^{1/h} = \lim(1 + hy)^{1/h} = \lim(1 + k)^{y/k}, \tag{10}$$

if $k = hy$; and either of these expressions may be denoted, concurrently, by the symbol e^y or $\exp(y)$. This is in fact Schlömilch's well known method of introducing the function e^y;* and that one of the most satisfactory current explanations of e^y as a limit should include the prior introduction of the function $(x^h - 1)/h$ is, I think, a circumstance which gives strong support to the analytic order now advocated.

We shall now see that the use of (1) as the definition of $\log(x)$ is particularly effective when x is complex. Let $x = r(\cos\theta + \iota\sin\theta)$, taking θ between $+\pi$ and $-\pi$. Observing that $\lim r^h = 1$, that $\lim\cos h\theta = 1$, and that $\lim\sin h\theta/h\theta = 1$, and remembering that $(\cos\theta + \iota\sin\theta)^h = \cos h\theta + \iota\sin h\theta$, we have

$$\begin{aligned}\log(x) &= \lim[r^h(\cos h\theta + \iota\sin h\theta) - 1]/h \\ &= \lim(r^h - 1)/h + \lim \iota\theta\sin h\theta/h\theta \\ &= \log(r) + \iota\theta,\end{aligned} \tag{10}$$

a result hitherto obtained by inverse processes. If we permit the value of θ to go beyond the limits assigned, we shall have multiple values for $\log(x)$, corresponding to the usual statement of such values.

Nor must it be supposed that real values of $\log(x)$ are less intelligible when (1) is employed as a definition than when, as usual, $\log(x)$ is presented as "y, where $e^y = x$." We are familiar with e, regarded as the limit of $(1 + k)^{1/k}$ when

---

*Zeitschrift für Mathematik, III, 387; Algebraische Analyse, 5th ed., 35–89; see also Chrystal, Algebra, II, 79. Schlömilch employs the function $(x^h - 1)/h$], but not its known limit, $\log(x)$, in presenting e^y. On the contrary, he adheres to the current view of $\log(x)$ as "y, where $e^y = x$."

k tends towards 0; we know how that function increases, and how $(1-k)^{-1/k}$ diminishes, as k diminishes, both tending towards the same numerical limit between 2 and 3. We shall now see that $\lim (x^h - 1)/h$ is at least as simple an idea, at least as easy to comprehend and illustrate, as $\lim (1+k)^{y/k}$, its inverse.

Let us consider the functions $(x^h - 1)/h$ and $(1 - x^{-h})/h$, which for distinction let us call the upper fraction and the lower fraction respectively. Here x and h are positive, and h is commensurable and not greater than 1. These fractions are definite continuous functions of x, devoid of mystery, positive when $x > 1$, negative where $x < 1$, and becoming 0 when $x = 1$. When $x = \infty$, $x^h = \infty$, $x^{-h} = 0$; so that as x varies from 0 to ∞, the upper fraction varies from $-h^{-1}$ to ∞, and the lower fraction from $-\infty$ to h^{-1}, each passing through 0 when $x = 1$. The upper fraction is equal to the lower fraction multiplied by x^h, so that the smaller the value of h the less their difference. If, for example, $x = 4096$, and if certain values be assigned to h, we find corresponding values for the fractions as follows:

	$h=1.$	$h=\frac{1}{2}.$	$h=\frac{1}{3}.$	$h=\frac{1}{4}.$	$h=\frac{1}{6}.$	$h=\frac{1}{12}.$
$\dfrac{x^h-1}{h}$	4095	126	45	28	18	12,
$\dfrac{1-x^{-h}}{h}$	$\dfrac{4095}{4096}$	$\dfrac{63}{32}$	$\dfrac{45}{16}$	$\dfrac{7}{2}$	$\dfrac{9}{2}$	6.

A well known algebraic inequality,

$$(x^h - 1)/h > (x^{h_1} - 1)/h_1, \tag{11}$$

where $h_1 < h$ teaches us that as h diminishes the upper fraction diminishes in value. The following inequality, presumably new, is readily derivable from (11):

$$(1 - x^{-h})/h < (1 - x^{-h_1})/h_1. \tag{12}$$

This shows that as h diminishes the lower fraction increases in value. But neither of them, one diminishing and the other increasing, can pass the other, for they have the same limit, because their ratio is x^h, of which the limit is 1. This common limit is denoted by the symbol $\log(x)$. A simple, and probably novel, proof of (11) and (12) will be given later (Appendix B).

But we need not rest here. I showed in the earlier paper that since $\lim u^h = 1$, u being any function of x, we may modify (1) thus:

$$\log(x) = \lim u^h (x^h - 1)/h. \tag{13}$$

In particular, if $u = x^{-ah}$,

$$\log(x) = \lim (x^{h-ah} - x^{-ah})/h.^{*} \tag{14}$$

The upper fraction is, in the limit, that special case of (14) in which $a = 0$, and the lower fraction is that special case in which $a = 1$. A third important case, which I indicated at the same time, is that in which $a = \frac{1}{2}$, the limit of a remarkable function which we may call the central fraction:

$$\log(x) = \lim (x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h. \tag{15}$$

This central fraction is the geometric mean between the upper and lower fractions, and may be illustrated in connection with them by employing the same example as before, $x = 4096$:

	$h=1.$	$h=\frac{1}{2}.$	$h=\frac{1}{3}.$	$h=\frac{1}{4}.$	$h=\frac{1}{6}.$	$h=\frac{1}{12}.$
$\dfrac{x^h - 1}{h}$	4095	126	45	28	18	12,
$\dfrac{x^{\frac{1}{2}h} - x^{-\frac{1}{2}h}}{h}$	$\dfrac{4095}{64}$	$\dfrac{63}{4}$	$\dfrac{45}{4}$	$7\sqrt{2}$	9	$6\sqrt{2}$,
$\dfrac{1 - x^{-\frac{1}{2}h}}{h}$	$\dfrac{4095}{4096}$	$\dfrac{63}{32}$	$\dfrac{45}{16}$	$\dfrac{7}{2}$	$\dfrac{9}{2}$	6.

The limit to which all three fractions converge is $8.317 = \log 4096$, and, as might be expected, the central fraction gives, for any small value of h, by far the best approximation of the three. The fact that

$$(x^h - 1)/h > (x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h > (1 - x^{-h})/h \tag{16}$$

may be shown at once from the existence of the factor $x^{\frac{1}{2}h}$, since x and h are both positive, and all three fractions are positive or negative simultaneously. I have found, however, and shall prove (Appendix C), that, in all cases wherein $x > 1$,

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h > (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1, \tag{17}$$

h_1 being smaller than h; and also that, when $0 < x < 1$,

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h < (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1. \tag{18}$$

Taking (17) with (12), therefore, we perceive that, when $x > 1$, the central fraction diminishes in value with h, while the lower fraction increases, so that their common limit log (x) lies between any value of the central fraction and any

---

[*] See also "Algebraic Proof of a Certain Series," *ante* p.

value of the lower fraction. On the other hand, when $0 < x < 1$, taking (18) with (11), we perceive that the central fraction increases algebraically (diminishing numerically because negative) when h diminishes, while the upper fraction diminishes algebraically (increasing numerically), so that their common limit $\log(x)$ lies between any value of the central fraction and any value of the upper fraction.

We are thus enabled, by means of the definition $\log(x) = \lim(x^h - 1)/h$, to obtain a clear idea of the real nature of a logarithm (impossible to obtain from the customary "y, where $x = e^y$"), as well as to deduce the usual properties of logarithms with the utmost ease. Employing this definition concurrently with either of the usual definitions of $\exp(y)$, whether defined as a limit or as a series, we may thus obtain a full and comprehensive view of the whole subject of logarithms and exponentials; and indeed, as we have seen, the readiest method of connecting the limit and the series expressing $\exp(y)$ is to resort to the limit which expresses $\log(x)$.

Perhaps the reader is still naturally reluctant to admit that any mode of presenting logarithms can take the place of that afforded by the analogy of common logarithms, the meaning of which is represented by the equation $10^{\lambda(n)} = n$. But the principle of concurrent definitions permits us to retain all rays of light derived from every quarter. The equation $\log(x) = y$, where $e^y = x$, is a truth which we may always use to the extent of its worth. Yet common logarithms were not originally discovered or computed by means of what seems to us the simple equation $10^{\lambda(n)} = n$. If we multiply both sides of (7) by k, a constant, we have, if $\phi(x) = k\log(x)$, $\phi(xu) = \phi(x) + \phi(u)$. That is to say, the logarithmic property which facilitates multiplication by prepared tables is possessed by any function $k\log(x)$ as well as by $\log(x)$. In Napier's tables, $k = -10{,}000{,}000$, and his "artificial numbers" (such was the circuitous route by which he approached the subject) contained another constant, say c, so that each of them might be represented by the formula $k\log(z) + c$. They could therefore be used only under restrictions. Subsequently, Briggs and Napier between them devised, and Briggs carried out, the idea of computing $p = k\log(z)$, where $k = 1/\log(10)$, the relation $z = 10^p$ affording unique advantages in practice. That this relation exists—or, more generally, $z = x^p$, where we define p as $k\log(z)$ and k as $1/\log(x)$—follows at once from (8); for $\log(x^p) = p\log(x) = \log(z)\log(x)/\log(x) = \log(z)$; that is, $x^p = z$. It is a matter of historic interest that Briggs's computation of the "common logarithm" of 2 was effected precisely upon the lines just indicated. He calculated $\log(2)$ and $\log(10)$ separately by the formula $\log(x) = \lim (x^h - 1)/h$, taking h very small by many extractions of square roots, and then obtained the "common logarithm" of 2 by multiplying $\log(2)$ by the modulus $1/\log(10)$.*

Reverting to our first list of possible definitions of $\log(x)$ and e^y, numbered (1) to (6), we find (2) still remaining to be discussed. The well-known series

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots\ldots = \psi(z), \tag{19}$$

may be used, as suggested in (2), taking $z = x - 1$ as the definition of $\log(x)$ whenever the series is real and summable, say when $-1 < z \overset{=}{<} 1$. If, in all other real cases, we define $\log(x)$ to be $-\log(x^{-1})$, I shall prove (Appendix D) that the series for $\log(x)$ added to the series for $\log(u)$ forms a sum equivalent to the series for $\log(xu)$, from which, as before, the essential properties of logarithms will follow. This is, in effect, accomplishing for the logarithmic series what is accomplished for the exponential series when the series for e^y and that for e^w are multiplied together to form a product equivalent to the series for e^{y+w}, according to Cauchy's well-known method. The same proof holds good for all cases in which x is not a real quantity (say either a complex quantity or a symbol of operation), and in which the several series involved are respectively interpretable.

Having once established that $\log(x) = \lim (x^h - 1)/h$, we shall find, on the one hand, that this definition of $\log(x)$ produces immediately, and with the greatest ease, the logarithmic series, if we substitute the binomial series $(1+z)^h$ for x^h, so that the binomial series forms a connecting link between the limit-definition and the series-definition of $\log(x)$. On the other hand, a different connecting link may be found in the limit customarily employed to define e^x. Assuming that we know that $\phi_2(x) + \phi_2(u) = \phi_2(xu)$, where, as in (2), $\phi_2(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \ldots$, and therefore that $\phi_2(x^n) = n\phi_2(x)$, it follows that $\phi_2(x) = \lim (x^h - 1)/h$, called $\log(x)$. For, let us suppose $\phi_2(x) = \log(x) . f(x)$, where $f(x)$ is some unknown function. Then $\phi_2(x^n) = \log(x^n) . f(x^n) = n\phi_2(x) = n \log(x) . f(x)$, and we know by (8) that $\log(x^n) = n \log(x)$. Dividing one of these by the other, we have $f(x^n) = f(x)$, so that $f(x)$ is independent of the value of x, since x^n may have any value; hence $f(x)$ is a constant, say c, so that $\phi_2(x) = c \log(x)$. Therefore

---

* Encyc. Brit.: "Logarithms."

$$c \log(1 + hz) = hz - \tfrac{1}{2} h^2z^2 + \ldots ,$$

and $c \lim \log(1 + hz)/h = \lim(z - \tfrac{1}{2} hz^2 + \ldots) = z$. But $\lim \log(1 + hz)/h = \log e^z = z$, whence $c = 1$, and

$$\log(1 + z) = z - \tfrac{1}{2} z^2 + \tfrac{1}{3} z^3 - \ldots . \tag{20}$$

Although the definition of $\log(x)$ as a series is by no means so general in its nature or so satisfactory as the series-definition of e^y, the consideration of $\log(x)$ from this point of view will necessarily add to the broadness and clearness of our knowledge of the function. Indeed, for mere intelligibility, it may be remarked that such an expression, for example, as $\log(\tfrac{3}{2}) = \tfrac{1}{2} - \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 - \ldots = 0.405$ nearly, is really easier for the mind to apprehend than $\log(\tfrac{3}{2}) = \lim(\tfrac{3}{2}^h - 1)/h$, or than "$y$, where $e^y = \tfrac{3}{2}$."

It is not difficult to expand the logarithm of a complex quantity by means of the logarithmic series. If $x = u + \iota v$ and $u = \pm \bmod u$, let $v/u = \pm x$, so that $x = (\pm 1 + \iota z).\bmod u$, and $\log(x) = \log(\bmod u) + \log(\pm 1 + \iota z)$. Here mod u is real and positive, and its logarithm may therefore be expressed by a series. As regards $\log(\pm 1 + \iota z)$, the case $\log(1 + \iota z)$, where $z^2 \leqq 1$, is well known, and the expansion is obtainable directly. If $z^2 > 1$, we may expand $\log(1 + \iota z)$ by taking y such that $(1 + \iota y)^2 = m(1 + \iota z)$, where m is real and positive, and here $y^2 < 1$, so that $\log(1 + \iota z) = 2\log(1 + \iota y) - \log m$, which may be expressed in a series. Again, if the sign be negative, we may take u such that $(1 + \iota u)^2 = n(-1 + \iota z)$, where n is real and positive, and $\log(-1 + \iota z)$ may be expressed as $2\log(1 + \iota u) - \log n$. The expression $\pm 1 + \iota z$ may be taken to represent a point in one of two perpendicular lines tangent to a circle whose radius is 1, in which case the unreal part of the logarithm represents the length of the arc cut off by a line from the point to the centre; and the unreal part of the series obtained is in the usual form of the series for an arc in terms of its tangent. That $\arctan x = \tfrac{1}{2} \iota \log[(1 - \iota x)/(1 + \iota x)]$ is known.

Appendix A.—If $\phi(x)$ be a function such that $\phi(x) + \phi(u) = \phi(xu)$, then $\phi(x^n) = n\phi(x)$. For, $\phi(xuv) = \phi(xu) + \phi(v) = \phi(x) + \phi(u) + \phi(v)$, and similarly, if there are n such quantities, $\phi(xu \ldots w) = \phi(x) + \phi(u) + \ldots + \phi(w)$.

Taking $u, v, \ldots . w$ severally equal to x, it follows that, when n is any positive integer,

$$\phi(x^n) = n\phi(x). \tag{21}$$

Since $\phi(x) + \phi(1) = \phi(x)$, $\phi(1) = 0$, and $\phi(x) + \phi(x^{-1}) = \phi(1) = 0$; whence $\phi(x^{-1}) = -\phi(x)$, so that, writing x^{-1} for x in (21), we have $\phi(x^{-n}) = -n\phi(x)$, so that (21) is true when n is a negative integer. That it is also true when n is fractional, say when $n = p/q$, may be shown by writing $x^{1/q}$ for x and q for n, whence $\phi(x) = q\phi(x^{1/q})$; then, bearing this in mind, and again writing $x^{1/q}$ for x in (21), but p for n, we have $\phi(x^{p/q}) = p\phi(x^{1/q}) = p/q\,\phi(x)$. The symbol x is not restricted here to real values; it may be complex, or it may be a symbol of operation. On the other hand, n is real and commensurable.

Appendix B.—If x, q and r are real and positive quantities, the following inequality is always true:

$$qx^q(x^r - 1) > (x - 1)\,qrx^q > r(x^q - 1). \tag{22}$$

To prove this when q and r are integers, we have only to recollect that $x^m - 1 = (x - 1)(x^{m-1} + x^{m-2} + \ldots . + x + 1)$; for $x^{r-1} + x^{r-2} + \ldots . + 1 >< r$, and $qx^q >< x^{q-1} + x^{q-2} + \ldots . + 1$, according as $x >< 1$; that is, according as the several members of (22) are all positive or all negative by reason of their common factor $x - 1$, the effect of a negative factor being to reverse the sign of inequality. Any such inequality proved true for integral exponents is necessarily true for fractional exponents, since we are at liberty to write $x^{1/p}$ for x. If we add $qx^q - q$ to both sides of (22), we have $q(x^{q+r} - 1) > (q + r)(x^q - 1)$; or, if we write h for $q + r$, and h_1 for q, and divide throughout by hh_1,

$$(x^h - 1)/h > (x^{h_1} - 1)/h_1. \tag{23}$$

On the other hand, if we divide (22) throughout by x^{q+r} and add $r - rx^{-r}$ to each side, we have $(q + r)(1 - x^{-r}) > r(1 - x^{-q-r})$; or, if we write h for $q + r$ and h_1 for r, and divide throughout by hh_1,

$$(1 - x^{-h_1})/h_1 > (1 - x^{-h})/h. \tag{24}$$

In each of these results h_1 is less than h. The inequality (23) is, as already stated, well known, and is recognized as highly important. (Cf. Chrystal, Algebra, II, 42–45.)

Appendix C.—The following inequality will be found more general and, so to speak, closer than the one (22) just discussed:

$$q(x^{q+r}-x^{-q-r}) >< (q+r)(x^q-x^{-q}), \tag{25}$$

according as $x >< 1$. In fact, it includes (22) as a special case. For, multiplying both sides of (25) by x^{q+r}, and writing $x^{\frac{1}{2}}$ for x, we have, according as $x >< 1$,

$$q(x^{q+r}-1) >< (q+r)(x^q-1)\,x^{\frac{1}{2}r}. \tag{26}$$

When $x > 1$, $x^{\frac{1}{2}r} > 1$, so that we have from (26)

$$q(x^{q+r}-1) > (q+r)(x^q-1),$$

which is merely (22) transposed. When $x < 1$, $x^{\frac{1}{2}r} > x^r$, so that we have from (26), changing signs,

$$q(1-x^{q+r}) > (q+r)(1-x^q)\,x^r,$$

whence, transposing,

$$rx^r(x^q-1) > q(x^r-1),$$

which is the same as (22), with q and r written each for the other. To demonstrate (25), we may remark that (m being an integer not greater than q), when $x >< 1$, $x^{2q-m+1} >< 1$, and therefore, multiplying both sides by x^m-1 (negative when $x < 1$) and transposing, we have, for all values of x, the inequality $x^{2q+1}+1 > x^{2q-m+1}+x^m$. Summing q such inequalities, in which m has successively all integral values from 1 to q inclusive, we obtain

$$q(x^{2q+1}+1) > x^{2q}+x^{2q-1}+\ldots\ldots+x^2+x.$$

Multiplying both sides by $x-1$ (negative when $x < 1$), we have now

$$q(x^{2q+2}+x-x^{2q+1}-1) >< x^{2q+1}-x,$$

or by transposition, after dividing throughout by x^{q+1},

$$q(x^{q+1}-x^{-q-1}) >< (q+1)(x^q-x^{-q}), \tag{27}$$

according as $x >< 1$; so that (25) is true when $r = 1$, for any positive integral value of q. Again, if (25) is true for any given integral value of r, it may be shown to be true for the value next greater, say $r+1$. For, if we write $(q+1)$ for q in (25) and multiply both sides by $q/(q+1)$, we have

$$\begin{aligned} q(x^{q+r+1}-x^{-q-r-1}) &>< (q+r+1)\,q(x^{q+1}-x^{-q-1})/(q+1) \\ &>< (q+r+1)(x^q-x^{-q}), \end{aligned}$$

since, by (27), $q(x^{q+1}-x^{-q-1})/(q+1) >< x^q - x^{-q}$. Hence, if (25) is true for any given value of r, it is true for the next higher value $r+1$; and since it is true for $r=1$, it is true for $r=2$, and so on for all other integral values. Proof for fractional exponents is to be supplied by writing $x^{1/p}$ for x. If, in (25), $\frac{1}{2}h$ be written for $q+r$, and $\frac{1}{2}h_1$, a smaller quantity than $\frac{1}{2}h$, for q, and both sides be divided by $\frac{1}{2}hh_1$, the general inequality takes this form:

$$(x^{\frac{1}{2}h} - x^{-\frac{1}{2}h})/h >< (x^{\frac{1}{2}h_1} - x^{-\frac{1}{2}h_1})/h_1, \tag{28}$$

according as $x >< 1$. In this paragraph, as in the preceding, all the quantities concerned are real and positive, and h and h_1 are commensurable.

APPENDIX D.—If the series $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots.$ be susceptible of interpretation, let us denote it by $\psi(z) = \phi_2(x)$, where $x = 1 + z$. It is to be proved that the sum of two such series, say $\phi_2(x) + \phi_2(u)$, is equal to $\phi_2(xu)$, provided the latter be also interpretable; or, if $w = u - 1$, that

$$\psi(z) + \psi(w) = \psi(zw + z + w). \tag{29}$$

The second member is a series of positive integral powers of the expression $z + w(1+z)$. Expanding these powers by the binomial theorem, assumed known for positive integral exponents, and separating the series forming the coefficient of $w^n(1+z)^n$, then expanding $(1+z)^n$ and multiplying the result by the coefficient just separated, and finally separating from the product the series forming the coefficient of $x^m w^n$, we find it to be, for all values of m and n greater than 0, the following expression multiplied by $(-1)^{n-1}/m!$:

$$(n-1)^{n-1)} - mn^{m-1)} + \ldots (-1)^r m^{r)} (n+r-1)^{n-1)}/r! \ldots (-1)^m (n+m-1)^{n-1)},$$

where $x^{k)}$ represents $x(x-1)(x-2)\ldots.(x-k+1)$. This expression is equal to 0, by a well-known theorem in finite differences, of which I have given an algebraic proof in a preceding paper ("Algebraic Proof of a Certain Series," *ante*, p. —). Therefore all terms in $x^m w^n$, that is to say, all terms which contain both x and w, vanish. The terms remaining, which contain z alone and w alone, are respectively $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \ldots. = \psi(z)$, and $w - \frac{1}{2}w^2 + \frac{1}{3}w^3 - \ldots. = \psi(w)$, so that (29) is proved true. The proof thus given is valid for real quantities when the three quantities concerned, namely, z, w and $zw + z + w$ are such that no one of them is greater than 1 and all are greater than -1. In such cases

x, u and xu are all greater than 0 and not greater than 2. Within these limits, therefore,

$$\phi_2(x) + \phi_2(u) = \phi_2(xu), \tag{30}$$

by (29), since $xu - 1 = (1 + z)(1 + w) - 1 = zw + z + w$. If, for any value of x greater than 2, we define $\phi_2(x)$ to be the series denoted by $-\phi_2(1/x)$, the property shown in (30) will still hold true. For, when x and u are both greater than 1, $\phi_2(1/x) + \phi_2(1/u) = \phi_2(1/xu)$, and on changing signs we have (30). If $x > 1$, $u < 1$, $xu < 1$, we have $\phi_2(xu) + \phi_2(1/x) = \phi_2(u)$, or $\phi_2(u) + \phi_2(x) = \phi_2(xu)$ by transposition. Or, if $x > 1$, $u < 1$, $xu > 1$, $\phi_2(1/xu) + \phi_2(u) = \phi_2(1/x)$, which gives the same result. A portion of this paragraph has been anticipated in my earlier paper, "An Essay on the Calculus of Enlargement," *American Journal of Mathematics*, II, 122, 123.

A Pair of Curves of the Fourth Degree and their Application in the Theory of Quadrics.

By H. B. Newson.

The curves in question are derived from the ellipsoid as follows: Let an ellipsoid be intersected by a series of planes forming an axial pencil, the axis of the pencil being perpendicular to the plane of the greatest and medium axes of the ellipsoid. The locus of the foci of the series of plane sections of the ellipsoid is a plane curve which I shall call throughout the Locus of Foci. The directrices of the same conic sections form a cylindrical surface whose section by the plane of the locus of the foci I shall call the Locus of Directrices. The equations of the curves are obtained as follows:

Let a, b, c be the axes of the ellipsoid in order of magnitude, and let its equation be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. The plane $z=0$ cuts from the ellipsoid a conic called the given conic, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and from the axial pencil a flat pencil whose vertex may be represented by (m, n). Each ray of this pencil is the transverse axis of the conic cut from the ellipsoid by the plane corresponding to the ray. The locus of centres of these sections is plainly a conic similar to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, and similarly placed, passing through the point (m, n) and the centre of the given conic. Transform the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to new axes parallel to the old and having the point (m, n) for origin, and then to polar coordinates. The equation of the given conic finally becomes

$$\left.\begin{aligned} r = {} & \frac{m\,(1-e^2)\cos\theta+n\sin\theta}{1-e^2\cos^2\theta} \\ & \pm\frac{\sqrt{(b^2-m^2(1-e^2)-n^2)(1-e^2\cos\theta)+(m\,(1-e^2)\cos\theta+n\sin\theta)^2}}{1-e^2\cos^2\theta}. \end{aligned}\right\}(1)$$

The rational part of this value of r denotes the radius vector of the locus of centres; and the radical part, the semi-transverse axes of the series of conic sections. If A represents the semi-transverse axis of a conic and E the eccentricity, then AE is the distance from the focus to the centre. But E in this case is $= \sqrt{e_2^2 \cos^2\theta + e_3^2 \sin^2\theta}$ (see Annals of Mathematics, Vol. V, page 5). Hence if we multiply the radical part of (1) by the value of E, the result is the radius vector of the locus of foci. Thus the polar equation of the curve sought is

$$\left.\begin{aligned} R = {} & \frac{m(1-e_1^2)\cos\theta + n\sin\theta}{1 - e_1^2\cos^2\theta} \\ & \pm \frac{\sqrt{\begin{aligned}&\{(b^2 - m^2(1-e_1^2) - n^2)(1 - e_1^2\cos^2\theta) \\ &\qquad + (m(1-e_1^2)\cos\theta + n\sin\theta)^2\}(e_2^2\cos^2\theta + e_3^2\sin^2\theta)\end{aligned}}}{1 - e_1^2\cos^2\theta} \end{aligned}\right\} \quad (2)$$

Transforming this back to rectangular coordinates by writing $\sqrt{x^2 + y^2}$ for R, $\frac{x}{\sqrt{x^2+y^2}}$ for $\cos\theta$, and $\frac{y}{\sqrt{x^2+y^2}}$ for $\sin\theta$, and remembering that $e_1^2 = \frac{a^2 - b^2}{a^2}$, $e_2^2 = \frac{a^2 - c^2}{a^2}$, $e_3^2 = \frac{b^2 - c^2}{b^2}$, we get

$$\left.\begin{aligned} &\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2} + \frac{m^2}{a^2} + \frac{n^2}{b^2} - 1\right)(x^2 + y^2) \\ &\qquad + c^2\left\{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - \left(\frac{my - nx}{ab}\right)^2\right\} = 0. \end{aligned}\right\} \quad (3)$$

Hence the locus of foci is a curve of the fourth degree.

The equation of the locus of directrices is obtained in similar manner. Remembering that the distance from the centre to directrix of a conic is $\frac{A}{E}$, we have

$$\left.\begin{aligned} R = {} & \frac{m(1-e_1^2)\cos\theta + n\sin\theta}{1 - e_1^2\cos^2\theta} \\ & \pm \frac{\sqrt{(b^2 - m^2(1-e_1^2) - n^2)(1 - e_1^2\cos^2\theta) + (m(1-e_1^2)\cos\theta + n\sin\theta)^2}}{(1 - e_1^2\cos^2\theta)\sqrt{e_2^2\cos^2\theta + e_3^2\sin^2\theta}}. \end{aligned}\right\} \quad (4)$$

In rectangular coordinates this becomes

$$\left.\begin{aligned} &\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2} + \frac{m^2}{a^2} + \frac{n^2}{b^2} - 1\right)(x^2 + y^2) \\ &\qquad = c^2\left\{\left(\frac{mx}{a^2} + \frac{ny}{b^2}\right)^2 + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2mx}{a^2} - \frac{2ny}{b^2}\right)\right\}. \end{aligned}\right\} \quad (5)$$

Hence the locus of directrices is also a curve of the fourth degree.

The point (m, n) is evidently a double point on each curve. The curves have double contacts with each other and with the given conic. The form of each curve varies greatly with the position of the point (m, n), which may be any point in the plane. We wish now to find the envelope of the loci of foci when the point (m, n) varies. Writing $(x + m)$ for x and $(y + n)$ for y in (3), the centre of the given conic becomes the origin. Equation (3) becomes

$$\left.\begin{aligned}\{(x + m)^2 + (y + n)^2\}\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right\}&\\ + c^2\left\{\frac{(x + m)^2}{a^2} + \frac{(y + n)^2}{b^2} - \frac{(my - nx)^2}{a^2b^2}\right\} &= 0.\end{aligned}\right\}(6)$$

Arranging this according to powers of m and equating to zero the discriminant with respect to m, we have after reduction :

$$\left[(y + n)^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \frac{c^2}{a^2b^2}(x^2 + y^2 - a^2 - b^2 + c^2)\right\}\right] = 0. \quad (7)$$

The squared factor is the locus of nodes, the given conic and the focal conic, $\dfrac{x^2}{a^2 - c^2} + \dfrac{y^2}{b^2 - c^2} = 1$, are envelopes.

By this method the equation of the focal conic is obtained without a previous knowledge of its properties, and might serve as a starting point for the whole theory of focal conics.

In exactly the same manner it can be shown that the envelopes of the loci of directrices are the given conic and the dirigent* conic (which is the polar reciprocal of the focal conic with respect to the given conic). With the origin at (m, n), the maximum and minimum radii vectores of the locus of foci are those to the points of contact with the focal conic.

It is not intended in this paper to discuss fully the properties of these interesting curves, but rather to make use of them in the investigation of certain properties of quadrics.

If the point (m, n) be at infinity, the locus of foci reduces to a conic. This is a well-known result. It may be shown as follows: Let $m = \infty$, $n = \infty$,

---

*So called by MacCullagh, see Proceedings Royal Irish Academy.

$\frac{n}{m} = \tan\theta$ in equation (6). Whence

$$\left.\begin{aligned}&\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\\&=\frac{c^2}{a^2b^2(1+\tan^2\theta)}\left(y-x\tan\theta+\sqrt{a^2\tan^2\theta+b^2}\right)\left(y-x\tan\theta+\sqrt{a^2\tan^2\theta+b^2}\right)\end{aligned}\right\}\tag{8}$$

which is the equation of a conic passing through the points where the two lines on the right cut the given conic. But these lines are parallel tangents to the given conic, and hence the two conics have double contact, the chord of contact being the diameter of the given conic conjugate to the direction of the point (m, n).

But equation (8) may also be written in the form

$$\left.\begin{aligned}&(1+\tan^2\theta)\left(\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}-1\right)\\&=\frac{-c^2}{(a^2-c^2)(b^2-c^2)}\left(x+y\tan\theta+\sqrt{(a^2-c^2)+(b^2-c^2)\tan^2\theta}\right)\left(x+y\tan\theta-\sqrt{(a^2-c^2)+(b^2-c^2)\tan^2\theta}\right)\end{aligned}\right\}\tag{9}$$

which shows that the locus of foci has double contact also with the focal conic, for the two lines on the right-hand side are parallel tangents to the focal conic. These parallel tangents are perpendicular to the direction of (m, n), and hence the chord of contact is the diameter of the focal conic conjugate to the direction perpendicular to the direction of (m, n).

When the point (m, n) is at infinity, the locus of directrices is also a conic, a result which I think is new. Transforming equation (5) to the centre of the given conic as origin, it becomes

$$\left.\begin{aligned}&\{(x+m)^2+(y+n)^2\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\\&\quad=c^2\left\{\left(\frac{mx}{a^2}+\frac{ny}{b^2}\right)^2+\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2mx}{a^2}-\frac{2ny}{b^2}\right)\right\}\end{aligned}\right\}\tag{10}$$

Making as before $m=\infty$, $n=\infty$, and $\frac{n}{m}=\tan\theta$, this becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\frac{c^2}{1+\tan^2\theta}\left(\frac{x}{a^2}+\frac{y\tan\theta}{b^2}\right)^2, \tag{11}$$

which is a conic having double contact with the given conic and with the locus of foci (which is now a conic), the chord of contact being the common diameter to all three conics.

Equation (10) can also be written in the form

$$\left\{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)\right\}=-\frac{c^2}{1+\tan^2\theta}\left(\frac{x\tan\theta}{a^2}-\frac{y}{b^2}\right)^2. \quad (12)$$

The left-hand side of (12) is the equation of the dirigent conic; hence, the locus of directrices has double contact with the dirigent conic, the chord of contact being coïncident with the chord of contact of the locus of foci and the focal conic.

It is easily seen that the general equation and properties of these two curves, determined for the case of the ellipsoid, are also true when the quadric is a paraboloid or an hyperboloid.

If $b = c$ in the equation of the ellipsoid, it becomes one of revolution around the axis of x. Many of the results already obtained are thereby greatly simplified. The focal conic degenerates into a right line joining the foci of a given conic. Since the loci of foci envelope the focal conic, it follows that the locus of foci always passes through the foci of the given conic, whatever may be the position of the point (m, n). This may also be shown analytically by putting $b = c$ and $y = 0$ in equation (6) and solving for x. We find $x = \pm ae$, a result independent of m and n.

Let the point (m, n) coincide with one of the foci of the given conic. Making $b = c$, $m = ae$ and $n = 0$ in equation (3), we have

$$(x^2+y^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2aex}{a^2}\right)=0. \quad (13)$$

Thus the locus of foci degenerates into a point circle at the origin and a conic. The equation of this conic referred to its own centre is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\varepsilon^2$; hence this conic is similar and concentric to the given one and has its vertices at the foci of the latter. The meaning of the point circle is that every section through the focus has that focus for a focus. It is clear that since we are here dealing with a surface of revolution, every section by a plane passing through the axis of x is a principal section. In each such section the locus of foci is a conic as above, and the assemblage of all these conics forms a surface of revolution similar and concentric to the given one, having its vertices at the foci of the latter.

The first part of this result is well known, and was first given by M. Chasles in the Mémoires de l'Académie de Bruxelles, Vol. V, page 33. The latter part seems not so well known, but I find it given by M. Olivier in Correspondance Mathématique et Physique, Vol. V, page 391.

In the case of quadrics of revolution, the locus of directrices also breaks up when the point (m, n) coincides with one of the foci of the given conic. Let $b = c$, $m = ae$, $n = 0$ in equation (5), which then becomes

$$\left\{ex\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-(x^2+y^2)\left(\frac{1+e^2}{a}\right)\right\}\left\{x+\frac{b^2}{ae}\right\}=0. \tag{14}$$

Thus the locus of directrices falls apart into a right line and a curve of the third degree. The right line is the directrix of the given conic corresponding to the focus on which the point (m, n) falls. This may be generalized for any principal plane of the quadric, and we have then the second part of Chasles' theorem concerning sections through a focus, viz. that one directrix of such a section lies in the polar plane of the focus through which the cutting planes pass. *The envelope of the other directrices is a surface of revolution of the third degree,* which has the plane $x = 0$ for an asymptote. This result is probably new.

Again let the point (m, n) be at the vertex of the given conic. Putting $b = c$, $m = a$, $n = 0$ in equation (5) we have

$$x^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2ax}{a^2}-\frac{b^2}{a^2-b^2}\right)=0. \tag{15}$$

Hence the locus of directrices falls apart into a right line and a conic. The equation of this conic transformed to its own centre is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{e^2}$. Hence it is similar and concentric to the given conic and has its foci at the vertices of the latter. The tangent plane at the vertex cuts from the quadric a point circle whose directrices are indeterminate. The case is satisfied by the line $x = 0$. As before these results may be generalized for all principal sections of the quadric. I shall now bring together and express in a convenient form the theorems of Chasles and Olivier and the results just developed.

Given two quadrics of revolution similar and concentric such that the foci of the one coincide with the vertices of the other. Any cutting plane passing through the vertex of the inner surface cuts from the outer surface a conic both of whose foci are on the inner surface, but one of which is always at the vertex of the inner surface through which the cutting plane passes. One directrix of the same conic is always in a fixed plane, the directrix plane of the outer surface. The other directrix is always tangent to a surface of revolution of the third degree. The same cutting plane cuts from the inner surface a conic both of whose foci are on a surface of revolution of the fourth degree and both of whose directrices are tangent to the outer surface.

When the given surface of the second degree is a cone, the section by the plane xy is a pair of intersecting lines. The only case of special interest is when the given point (m, n) falls on one of these lines. The locus of foci and locus of directrices each break up into the right line on which the point falls and a curve of the third degree. The locus of foci in this case once attracted considerable attention and was studied in detail by a group of Belgian mathematicians.

The curve was first studied by Quetelet in a paper entitled *Dissertatio de quibusdam locis geomet. nec non de Curvâ focali:* Gandavi, 1819. This paper I have not seen. In the Journal of the Royal Academy of Brussels I find the following entry for December 6, 1819:

"M. le Commandeur de Niewport a présenté au nom de M. Quetelet, professeur de mathématique à l'athénée de Bruxelles, un mémoire sur quelques nouvelles propriétés de la focale et sur la même courbe focale. * * * * * L'Académie avait chargé M. Quetelet de refondre ses deux mémoires en un seul; mais l'auteur ayant appris que M. Dandelin s'occupait d'un écrit sur le même sujet a cru renouncer à son travail, d'autant plus que celui de son ami ne laissait rien à desirer sur ce point."

Dandelin's paper is published in the Mémoires de l'Académie de Bruxelles, Vol. II. It deals only with the case of the right circular cone. A memoir by Van Rees in Quetelet's Correspondance Mathématique et Physique, Vol. VI, page 361, treats of the curve in both the principal sections of the elliptic cone.

In case of the elliptic cone the curve consists of two branches, each of which is tangent to a focal line of the cone. One branch touches the given line at the point (m, n) and the other branch has the other given line for an asymptote. If the cone is right circular, the locus of foci has a double point and is the "focale a nœud" of Quetelet recently discussed by Morley (*American Journal,* XI, page 307).

The locus of directrices in the case of the cone is a curve of the third degree whenever the locus of foci is of the third degree. The Belgian mathematicians who studied the properties of the locus of foci do not seem to have noticed the locus directrices. I add without proof a number of theorems on the focal properties of series of quadrics which may readily be proved by means of the methods and formulas already developed.

Given a series coaxial and concentric hyperboloids of revolution of one nappe all having the same circle of the gorge; if the axis of an axial pencil of planes is tangent to the circle of the gorge, the locus of foci of parabolic sections is a circle; the locus of directrices of the same sections is also a circle.

If the axis of an axial pencil of planes is perpendicular to the axial plane of a parabolic cylinder and tangent to the cylinder, the locus of foci of parabolic sections is a circle, so is also the locus of directrices.

Given a series of right circular cones on the same base, if the axis of an axial pencil of planes is tangent to the base circle, the locus of foci of parabolic sections is a circle; so is also the locus of directrices.

The eccentricity of the section of a quadric of revolution by a plane passing through a focus and containing any tangent to the central circular section is equal to the square of the eccentricity of the quadric.

If the axis of an axial pencil of planes is tangent to the base circle of a right circular cone, the distance from the centre to the foci of any section is equal to the distance from the centre of the section to the centre of the base circle. Given the same conditions as above, the perpendicular from the centre of the base circle on the parabolic section passes through the focus of the parabola.

A Note on Linear Transformation.

BY H. P. MANNING.

The following was suggested by a method which Professor Cayley has employed in his "Memoir on the Abelian and Theta Functions," in the *American Journal of Mathematics*, Vol. V.

Suppose we have $f(x_1, x_2, \ldots . x_k)$, a polynomial in k variables, and wish to transform it into a function of y's by the substitution

$$x_i = \sum_{j=1}^{j=k} \alpha_{ij} y_j \qquad (i = 1, 2, \ldots . k).$$

Represent it symbolically by a_x^n where

$$a_x = a_1 x_1 + a_2 x_2 + \ldots . + a_k x_k,$$

and let the transform be A_y^n, where

$$A_y = A_1 y_1 + A_2 y_2 + \ldots . + A_k y_k.$$

If we let δ_i denote the polar operation, namely,

$$\delta_i = \alpha_{1i} \frac{\partial}{\partial x_1} + \alpha_{2i} \frac{\partial}{\partial x_2} + \ldots . + \alpha_{ki} \frac{\partial}{\partial x_k},$$

then

$$A_i = \delta_i a_x,$$
$$A_i A_j = \delta_i a_x . \delta_j a_x = \tfrac{1}{2} \delta_i \delta_j a_x^2,$$

and in general,

$$A_i^a A_j^b \ldots . = \frac{1}{(a + b + \ldots .)!} \delta_i^a \delta_j^b \ldots . a_x^{a+b+\ldots .}.$$

Hence A_y^n may be written

$$\frac{1}{n!} (y_1 \delta_1 + y_2 \delta_2 + \ldots . + y_k \delta_k)^n a_x^n,$$

that is, f becomes

$$\frac{1}{n!}(y_1\delta_1 + y_2\delta_2 + \ldots\ldots + y_k\delta_k)^n f.$$

We can write down at once a formula for the coefficient of any term:

$$\text{coeff. of } y_i^a y_j^b \ldots\ldots = \frac{1}{a!\,b!\,\ldots\ldots}\,\delta_i^a\delta_j^b\ldots\ldots f \qquad (a + b + \ldots\ldots = n)\cdot$$

In each case there are n polar operations, and these will always introduce the same numerical factor, $n!$

The n polar operations remove all the x's and replace them by α's. Moreover, $\delta_i^n f$ is equivalent to f with all the x's replaced by α_{1i}, $\alpha_{2i}\ldots\ldots$, and we may in any particular case reduce the number of polar operations actually to be performed by a number equal to the highest exponent in the term. For example, to obtain $\delta_1^3\delta_5^{n-3}f$ we have only to perform the operation

$$\left(\alpha_{11}\frac{\partial}{\partial\alpha_{15}} + \alpha_{21}\frac{\partial}{\partial\alpha_{25}} + \ldots\ldots\right)^3 f(\alpha_{15}, \alpha_{25}, \ldots\ldots).$$

Example: Suppose we wish to transform the cubic

$$f = ax^3 + by^3 + cz^3 - 6lxyz$$

by the substitution

$$\begin{aligned} x &= \alpha_1 X + \alpha_2 Y + \alpha_3 Z, \\ y &= \beta_1 X + \beta_2 Y + \beta_3 Z, \\ z &= \gamma_1 X + \gamma_2 Y + \gamma_3 Z, \end{aligned}$$

we have

$$\begin{aligned} \delta_i &= \alpha_i\frac{\partial}{\partial x} + \beta_i\frac{\partial}{\partial y} + \gamma_i\frac{\partial}{\partial z}, \\ \delta_i f &= 3\,(a\alpha_i x^2 + b\beta_i y^2 + c\gamma_i z^2 - 2l\alpha_i yz - 2l\beta_i zx - 2l\gamma_i xy), \\ \delta_i\delta_j f &= 6\,(a\alpha_i\alpha_j x + \ldots\ldots - l\alpha_i\beta_j z - \ldots\ldots); \\ \therefore\ f &= (a\alpha_1^3 + b\beta_1^3 + c\gamma_1^3 - 6l\alpha_1\beta_1\gamma_1)\,X^3 + \ldots\ldots \\ &\quad + 3\,(a\alpha_1^2\alpha_2 + \ldots\ldots - 2l\beta_1\gamma_1\alpha_2 - \ldots\ldots)\,X^2 Y + \ldots\ldots \\ &\quad + 6\,(a\alpha_1\alpha_2\alpha_3 + \ldots\ldots - l\alpha_1\beta_2\gamma_3 - \ldots\ldots)\,XYZ. \end{aligned}$$

Some Theorems relating to Groups of Circles and Spheres.[*]

By Wm. Woolsey Johnson.

---

1. In the earliest of Cayley's published papers, and that which stands first in his collected works—"On a Theorem in the Geometry of Position," Cambridge Math. Journal (1841), Vol. II, p. 267—the theorem for the multiplication of two determinants (then new) was applied to Carnot's problem: To find the relation which exists between the distances of five points in space. The same method is applied to find the relation between the distances of four points in a plane, and the additional relation which exists when the points are on the circumference of a circle. The m^{th} row of the first of the determinants multiplied together is

$$x_m^2 + y_m^2, \quad x_m, \quad y_m, \quad 1,$$

where x, y are rectangular coordinates, and the n^{th} row of the second is

$$1, \quad -2x_n, \quad -2y_n, \quad x_n^2 + y_n^2;$$

so that the corresponding element of the product is

$$(x_m - x_n)^2 + (y_m - y_n)^2,$$

or the square of the distance between the points (x_m, y_m) and (x_n, y_n). Taking the same four points in each of the factor determinants, and adding to the first the fifth row

$$1, \quad 0, \quad 0, \quad 0,$$

and to the second the fifth row

$$0, \quad 0, \quad 0, \quad 1,$$

and to each a fifth column of zeros, the product is zero, and the required general relation is found. Again, if the four points lie on a circle, each factor determinant is zero without the additional row and column and the product is zero, giving the special relation.

---

[*] Some of the results contained in this paper were presented to the British Association at the Leeds meeting in 1890.

2. By an extension of the method the m^{th} row of the first determinant is taken as

$$x_m^2 + y_m^2 - r_m^2, \quad x_m, \quad y_m, \quad 1,$$

where r_m is the radius of a circle whose centre is (x_m, y_m), and

$$1, \quad -2X_n, \quad -2Y_n, \quad X_n^2 + Y_n^2 - R_n^2$$

as the n^{th} row of the other, so that the corresponding element of the result is

$$(x_m - X_n)^2 + (y_m - Y_n)^2 - r_m^2 - R_n^2,$$

which is defined as the relative power of the circles (x_m, y_m, r_m) and (X_n, Y_n, R_n), these circles belonging to two separate groups. Then, 1°, if there be five circles or six spheres in each group, the product or determinant of powers is equal to zero, and 2°, if there be four circles or five spheres in each group, the power determinant is the product of two determinants each of which depends upon one of the groups.

Mr. Lachlan has developed a great variety of geometrical consequences, derived principally from the first of these theorems, in his memoir "On Systems of Circles and Spheres," Phil. Trans. Vol. 177, (1886), p. 481. It is the object of the present paper to point out some other results derivable from the second theorem, and particularly to evaluate the power determinants for groups of smaller numbers of circles and spheres.

3. Let the four circles whose centres are $(x_1, y_1) \ldots\ldots (x_4, y_4)$ and whose radii are $r_1 \ldots\ldots r_4$ be denoted by a, b, c, d, and put

$$\begin{vmatrix} x_1^2 + y_1^2 - r_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 - r_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 - r_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 - r_4^2 & x_4 & y_4 & 1 \end{vmatrix} = \Delta(a, b, c, d); \qquad (1)$$

then for another group of circles A, B, C, D we have

$$\begin{vmatrix} 1 & -2X_1 & -2Y_1 & X_1^2 + Y_1^2 - R_1^2 \\ 1 & -2X_2 & -2Y_2 & X_2^2 + Y_2^2 - R_2^2 \\ 1 & -2X_3 & -2Y_3 & X_3^2 + Y_3^2 - R_3^2 \\ 1 & -2X_4 & -2Y_4 & X_4^2 + Y_4^2 - R_4^2 \end{vmatrix} = -4\Delta(A, B, C, D).$$

Denoting the relative power of the circles a and A by (aA), etc., we have by multiplication

$$\begin{vmatrix} (aA) & (aB) & (aC) & (aD) \\ (bA) & (bB) & (bC) & (bD) \\ (cA) & (cB) & (cC) & (cD) \\ (dA) & (dB) & (dC) & (dD) \end{vmatrix} = -4\Delta(a, b, c, d)\,\Delta(A, B, C, D). \quad (2)$$

4. The relative power of two circles as (aA) is the square of the distance of their centres diminished by the sum of the squares of the radii. When the circles intersect it is $2Rr\cos\omega$, where ω is the internal angle of intersection, and *it vanishes for two circles which cut at right angles.* When one of the circles reduces to a point, it becomes what is usually called the power of the point relatively to the circle and vanishes for a point on the circumference. When both circles reduce to points, it is the square of their distance.

5. In equation (1) the elements in any row may be called the elements of the corresponding circle. For the circle altogether at an infinite distance the first element is infinitely greater than either of the others. Dividing out this infinite element, we may take for the elements of the circle at infinity

$$1, \quad 0, \quad 0, \quad 0.$$

To preserve equation (2), it is then obviously necessary to take unity as the power of any finite circle relatively to infinity, and zero as the power of infinity relatively to itself.

Again, if the circle become the straight line whose equation is

$$x\cos\alpha + y\sin\alpha = p_0,$$

putting ρ for the infinite distance of the centre from the origin, the elements are $\rho^2 - r^2$ or $(\rho + r)p_0$, $\rho\cos\alpha$, $\rho\sin\alpha$, 1, and dividing by the infinite radius of the circle we may take them to be

$$2p_0, \quad \cos\alpha, \quad \sin\alpha, \quad 0.$$

It follows that to preserve equation (2), the relative power of this line and the finite circle (X, Y, R) must be taken as $2p_0 - 2X\cos\alpha - 2Y\sin\alpha$, or twice the perpendicular from the centre of the circle to the line (as is indeed otherwise evident), the side of the line on which p is positive being regarded as the outside of the infinite circle. The power of infinity relative to a straight line must be taken as zero, and the relative power of two lines as $-2\cos(\alpha - \beta)$; or, since the side on which p is positive is taken as the outside, it is twice the cosine of the angle between the insides of the lines, agreeing with the expression $2rR\cos\omega$, which is now to be divided by the product of the radii of both circles.

6. Making the second group of points in equation (2) identical with the first, we have for the symmetrical determinant of mutual powers

$$\begin{vmatrix} -2r_1^2 & (ab) & (ac) & (ad) \\ (ab) & -2r_2^2 & (bc) & (bd) \\ (ac) & (bc) & -2r_3^2 & (cd) \\ (ad) & (bd) & (cd) & -2r_4^2 \end{vmatrix} = -4\Delta^2(a,\, b,\, c,\, d), \tag{3}$$

in which for (aa) is put its value $-2r_1^2$, etc.

It follows from this equation that the value of $\Delta(a, b, c, d)$ is independent of the position of the origin and axes.

Thus $\Delta(a, b, c, d)$ depends only upon the configuration of the four circles. If one of them, say a, is removed to infinity, Δ becomes infinite, but taking the reduced elements, as in §5, we have

$$\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \Delta(\infty,\, b,\, c,\, d) = \Delta(b,\, c,\, d),$$

or double the area of the triangle whose vertices are the centres of b, c and d.

7. The elements of the first column in the determinant Δ, equation (1), are the powers of the origin, O, with respect to the circles a, b, c, d; that is, in the notation adopted, (Oa), (Ob), (Oc) and (Od). Denoting the corresponding minors by $\alpha, \beta, \gamma, \delta$ we have

$$\alpha(Oa) + \beta(Ob) + \gamma(Oc) + \delta(Od) = \Delta(a,\, b,\, c,\, d) = \text{constant} \tag{4}$$

for all positions of O. Moreover, since, if we replace the first column of Δ by a column of units, we have

$$\alpha + \beta + \gamma + \delta = 0, \tag{5}$$

we may, by subtracting $(\alpha + \beta + \gamma + \delta)R^2 = 0$, replace $(Oa) \ldots . (Od)$ by $(Oa) - R^2 \ldots . (Od) - R^2$, the powers of a circle whose centre is O and radius R relative to $a \ldots . d$. Thus equation (4) expresses a linear relation between the powers of any circle with respect to four given circles.

8. If we eliminate $\alpha, \beta, \gamma, \delta$ and the constant between four equations of the form (4) (in which A, B, C, D respectively are put for O) and equation (5),

we shall have a relation between the powers which appear in equation (2), namely

$$\begin{vmatrix} (aA) & (aB) & (aC) & (aD) & 1 \\ (bA) & (bB) & (bC) & (bD) & 1 \\ (cA) & (cB) & (cC) & (cD) & 1 \\ (dA) & (dB) & (dC) & (dD) & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix} = 0, \qquad (6)$$

the same relation which is obtained for two groups of five circles, the fifth circle in each group being at infinity.

Values of Δ.

9. In equation (4), let O be the circle orthogonal to b, c and d; the equation reduces to

$$\Delta(a, b, c, d) = (Oa).\Delta(b, c, d); \qquad (7)$$

that is, the value of the Δ of four circles is *the power of any one of them relatively to the circle which is orthogonal to the other three, multiplied by the double area of the triangle whose vertices are the centres of these three circles.* In (7), $\Delta(b, c, d)$ is positive when the rotary direction bcd is positive.

If b, c, d are fixed circles, O is fixed, and the circles, a, which satisfy $\Delta(a, b, c, d) =$ constant are those which have a fixed power relatively to O, or, what is the same thing, those which cut orthogonally a certain fixed circle concentric with O.

10. It follows from equation (7) that Δ *vanishes* when a is orthogonal to O; that is, *when there exists a circle O which cuts all four circles orthogonally*, or, when the *four circles have a common radical centre.*

If the centres of b, c and d lie in a straight line, the factor $\Delta(b, c, d)$ vanishes, but $\Delta(a, b, c, d)$ does not vanish in this case, for O becomes a straight line, so that (Oa) contains an infinite factor, namely, the radius of the infinite circle O, the cofactor being, as in §5, twice the perpendicular upon the line O from the centre of a. The radius of a does not now enter the expression, so that a may be taken as a point. If b, c and d are also points, we readily derive a new expression for Δ in this case. For O is now the circle circumscribing the triangle bcd, and denoting its radius by R, we have $4R \times$ area = product of sides, therefore

$$\Delta(a, b, c, d) = 2Rp\Delta(b, c, d) = 4Rp \times \text{area} = p.bc.cd.db;$$

that is, *if b, c, d are points on a straight line, and a is at a distance p from it,*

$$\begin{vmatrix} 0 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \\ x_4^2+y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = p.bc.cd.db, \tag{8}$$

the first element being omitted because its corresponding minor vanishes by hypothesis.

11. When one or more of the circles a, b, c, d is replaced by ∞ or by a straight line, the elements being modified as in §5, the value of Δ is modified accordingly. Thus, as mentioned in §6, if ∞ is put for a, $\Delta(a, b, c, d)$ becomes $\Delta(b, c, d)$, the double area of the triangle of centres of the circles b, c, d. If, furthermore, one of the three circles becomes a straight line, the modified value of Δ is readily seen to be the projection upon this line of the distance between the centres of the two circles remaining finite.

12. If a becomes a straight line, b, c, and d remaining finite circles, the general value of Δ, equation (7), holds if we modify accordingly the meaning of (Oa) as explained in §5, namely, (Oa) is twice the perpendicular from the centre of O upon a. That is to say, given a straight line and three circles, we have

$$\begin{vmatrix} 2p_0 & \cos\alpha & \sin\alpha & 0 \\ x_1^2+y_1^2-r_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2-r_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2-r_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 2p \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \tag{9}$$

which gives an expression for p the perpendicular upon a given straight line from the centre of the circle orthogonal to three given circles.

If X and Y are the coordinates of the centre of O, we have in the notation employed

$$p = p_0 - X\cos\alpha - Y\sin\alpha,$$

which becomes X when $p_0 = 0$, $\cos\alpha = -1$, and becomes Y when $p_0 = 0$, $\sin\alpha = -1$; hence the coordinates of the centre of the circle orthogonal to $(x_1, y_1, r_1)(x_2, y_2, r_2)(x_3, y_3, r_3)$ are given by

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2 & y_1 & 1 \\ x_2^2+y_2^2-r_2^2 & y_2 & 1 \\ x_3^2+y_3^2-r_3^2 & y_3 & 1 \end{vmatrix} = 2X \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

$$\begin{vmatrix} x_1^2+y_1^2-r_1^2 & x_1 & 1 \\ x_2^2+y_2^2-r_2^2 & x_2 & 1 \\ x_3^2+y_3^2-r_3^2 & x_3 & 1 \end{vmatrix} = -2Y \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

13. If b as well as a becomes a straight line, the circle orthogonal to b, c, d has its centre on this line b, that is, at the intersection of this line with the radical axis of c and d. Let p denote the perpendicular from this point upon a, and q the projection on b of the distance between the centres of c and d. Then (Oa) becomes $2pR_1$ and $\Delta(b, c, d)$ becomes qR_2, where R_1 and R_2 are the infinite radii divided out in writing the modified form of Δ. Another expression for Δ arises from taking the finite circles to be a and b. Let M be the intersection of the lines c and d, and let A and B be the centres of the circles. Then O is the circle whose centre is at M and whose squared radius is $\overline{MB}^2 - r_2^2$. The power (Oa) is therefore $\overline{MA}^2 - \overline{MB}^2 - r_1^2 + r_2^2$, while $\Delta(b, c, d)$ when divided by the infinite factors becomes $\sin(\beta - \alpha)$. Thus we have the two expressions

$$\begin{vmatrix} x_1^2 + y_1^2 - r_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 - r_2^2 & x_2 & y_2 & 1 \\ 2p_0 & \cos\alpha & \sin\alpha & 0 \\ 2p_0' & \cos\beta & \sin\beta & 0 \end{vmatrix} = 2pq = (\overline{MA}^2 - \overline{MB}^2 - r_1^2 + r_2^2)\sin(\beta - \alpha), \quad (10)$$

where q is the projection of the distance between the centres of the circles upon one of the lines, p the perpendicular from the intersection of this line with the radical axis of the circles upon the other line, M the intersection of the lines, and A and B the centres of the circles.

14. When three of the four circles become straight lines, the fourth being finite, the value of $\Delta(a, b, c, d)$ is independent of the latter and becomes twice the product of the perpendicular upon one of the lines from the intersection of the other two into the sine of the angle between these two lines, or the product of the sides of the triangle formed by the three lines divided by twice the square of the radius of its circumscribing circle.

The Power Determinant for Groups of Four Circles.

15. The determinant of powers of two groups of circles is denoted by $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix}$, then equation (2) is

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D \end{pmatrix} = -4\Delta(a, b, c, d)\,\Delta(A, B, C, D).$$

We notice in the first place that the value of Π depends only upon the configuration of the two groups respectively, and not upon their relative position, so that, if $abcd$ and $ABCD$ are two given quadrilaterals, the value of $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D\end{pmatrix}$ is unchanged when the relative position of the quadrilaterals is altered.

The value of $\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D\end{pmatrix}$ vanishes when either Δ vanishes; that is, when either group has a common orthogonal circle (or common radical centre).

When the two groups are identical, we have

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ a, & b, & c, & d\end{pmatrix} = -4\Delta^2(a, b, c, d),$$

which is always negative. If a, b, c, d are points, we have, for the general value of the quantity which vanishes when the points are concyclic,

$$\begin{vmatrix} 0 & \overline{ab}^2 & \overline{ac}^2 & \overline{ad}^2 \\ \overline{ab}^2 & 0 & \overline{bc}^2 & \overline{bd}^2 \\ \overline{ac}^2 & \overline{bc}^2 & 0 & \overline{cd}^2 \\ \overline{ad}^2 & \overline{bd}^2 & \overline{cd}^2 & 0 \end{vmatrix} = -4(\overline{Oa}^2 - R^2)^2\Delta^2(b, c, d), \tag{11}$$

where Δ is the double area of the triangle bcd, O the centre and R the radius of its circumscribing circle.

The Power Determinant for Groups of Three Circles.

16. Now let a, b, c and A, B, C be any two groups of three circles each; take for d the circle orthogonal to A, B, C, and for D that orthogonal to a, b, c. (The centres are the radical centres of the groups and the radii may of course be imaginary.) Then, in equation (2), the powers (aD), (bD), (cD), (dA), (dB), (dC) vanish, and we have

$$(dD)\begin{vmatrix} (aA) & (aB) & (aC) \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4\Delta(a, b, c, d)\,\Delta(A, B, C, D);$$

but, by equation (7),

$$\Delta(a, b, c, d) = -(Dd).\Delta(a, b, c), \qquad \Delta(A, B, C, D) = -(dD).\Delta(A, B, C),$$

hence

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = \begin{vmatrix} (aA) & (aB) & (aC) \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4\Delta(a, b, c)\,\Delta(A, B, C)\,(dD).^{*} \qquad (12)$$

That is, the power determinant of two groups of three circles each is — 16 *times the product of the areas of the triangles formed by the centres of the groups and the relative power of the circles respectively orthogonal to the two groups.*

It follows that *the power determinant of two groups of three circles each vanishes when the circles respectively orthogonal to the two groups cut each other orthogonally.*

17. If the two groups are identical we have

$$\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix} = \begin{vmatrix} -2r_1^2 & (ab) & (ac) \\ (ab) & -2r_2^2 & (bc) \\ (ac) & (bc) & -2r_3^2 \end{vmatrix} = 8\Delta^2(a, b, c)\,R^2, \qquad (13)$$

where R is the radius of the circle orthogonal to the group. This expression vanishes when $R = 0$; that is, when the three circles pass through a common point. It does not, however, vanish when $\Delta(a, b, c) = 0$ (that is, when the centres of the circles are in a straight line), because R is then infinite, unless the circles have a common radical axis, in which case R is indeterminate. Thus $\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix}$ vanishes only when the three circles have one common point, or two common points real or imaginary.

But, when the centres of a, b, c lie in a straight line, the determinant $\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix}$ admits of reduction. Denoting the sides of the triangle opposite the centres of a, b, c by d_1, d_2, d_3 respectively, and supposing $d_1 = d_2 + d_3$, we shall find

$$\Pi\begin{pmatrix} a, & b, & c \\ a, & b, & c \end{pmatrix} = 2\left[d_2\left(d_3^2 + r_1^2 - r_2^2\right) + d_3\left(d_2^2 + r_1^2 - r_3^2\right)\right]^2. \qquad (14)$$

---

[*] It is erroneously stated in Mr. Lachlan's memoir that

$$\left[\Pi\begin{pmatrix} 1, & 2, & 3 \\ 4, & 5, & 6 \end{pmatrix}\right]^2 = \Pi\begin{pmatrix} 1, & 2, & 3 \\ 1, & 2, & 3 \end{pmatrix}\Pi\begin{pmatrix} 4, & 5, & 6 \\ 4, & 5, & 6 \end{pmatrix}$$

(Phil. Trans., Vol. 177 (1886), p. 498). This would imply that the power determinant for groups of three circles depended only upon the configuration of the groups and not upon their relative position. This property belongs only to groups of four circles. See §15.

The quantity in brackets may also be written

$$d_2\,(d_1^2 + r_3^2 - r_2^2) - d_1\,(d_2^2 + r_3^2 - r_1^2)\,,$$

or symmetrically, if $d_1 + d_2 + d_3 = 0$,

$$-\,d_1d_2d_3 - d_1r_1^2 - d_2r_2^2 - d_3r_3^2.$$

18. If, in equation (12), the centres of a, b, c lie in a straight line, so that $\Delta\,(a,\,b,\,c)$ vanishes, $\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix}$ does not vanish, because (dD) is then infinite, its value being $2pR$, where p is the perpendicular from the radical centre of A, B, C upon the straight line, and R the infinite radius of the circle orthogonal to a, b, c. We have then

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = -\,8pR\Delta\,(a,\,b,\,c)\,\Delta\,(A,\,B,\,C).$$

Comparing equations (13) and (14) and disregarding sign, we have for the value of $R\Delta\,(a,\,b,\,c)$, when the centres lie in a straight line and their distances are so taken that $d_1 + d_2 + d_3 = 0$,

$$2R\Delta\,(a,\,b,\,c) = d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2. \tag{15}$$

Substituting, we have for this case

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 4p\Delta\,(A,\,B,\,C)(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2). \tag{16}$$

If the centres of A, B, C are also in a straight line, and ω is the angle between the lines, p becomes infinite and $= \rho\cos\omega$, where ρ is the radius of the orthogonal circle, and we have, for this case,

$$\left.\begin{aligned} \Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 2\cos\omega\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2) \\ (D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2)\,. \end{aligned}\right\} \tag{17}$$

19. Let a, b, c be circles passing through a common point, and let A, B, C be points, then D is the common point of a, b, c, and d is the circle passing through A, B, C; hence, if this circle passes through the common point, the power determinant vanishes. If any point on d, say D_1, be joined with the common point D, its powers relatively to a, b and c are proportional to the distances from D_1 to the other intersections of this line with a, b and c respectively. Thus, if four circles

pass through a common point, and three straight lines through this point meet the circles again in A_1, B_1, C_1, D_1, etc., we shall have

$$\begin{vmatrix} D_1A_1 & D_1B_1 & D_1C_1 \\ D_2A_2 & D_2B_2 & D_2C_2 \\ D_3A_3 & D_3B_3 & D_3C_3 \end{vmatrix} = 0. \qquad (18)$$

20. When one or more of the circles are replaced by lines, the powers being taken as in §5, the value of Π is modified by the omission of infinite factors and it is easy to make the corresponding modification of the second member of equation (12). Thus if a be a straight line, P_1, P_2, P_3 being the perpendiculars from the centres of A, B and C, the theorem is

$$\begin{vmatrix} 2P_1 & 2P_2 & 2P_3 \\ (bA) & (bB) & (bC) \\ (cA) & (cB) & (cC) \end{vmatrix} = -4q\Delta(A,\, B,\, C).(dD), \qquad (19)$$

where q is the projection upon the line a of the line joining the centres of b and c, taken with the sign which would be given to $\Delta(a,\, b,\, c)$.

Again, if there be a line in each group, we have

$$\begin{vmatrix} 2\cos\omega & 2P_2 & 2P_3 \\ 2p_2 & (bB) & (bC) \\ 2p_3 & (cB) & (cC) \end{vmatrix} = -4Qq(dD), \qquad (20)$$

where ω is the angle between the lines on that side of each on which the perpendiculars have like signs.

If there be two lines in one group, say A and B are lines, we have, dropping the factor 4,

$$\begin{vmatrix} p_1 & p'_1 & (aC) \\ p_2 & p'_2 & (bC) \\ p_3 & p'_3 & (cC) \end{vmatrix} = -\Delta(a,\, b,\, c)\sin\omega\,(dD), \qquad (21)$$

where ω is the angle between the lines and the signs of the perpendiculars are so taken that $\Delta(A,\, B,\, C)$ would be positive.

If there be two lines in each group, we have in like manner

$$\begin{vmatrix} 2\cos(aA) & 2\cos(aB) & 2P_1 \\ 2\cos(bA) & 2\cos(bB) & 2P_2 \\ 2p_1 & 2p_2 & (cC) \end{vmatrix} = -4\sin(ab)\sin(AB).(dD), \qquad (22)$$

where $\cos(aA)$ is the cosine of the angle between the lines a and A, etc.

If one group, say A, B, C, consists of three straight lines, the theorem is concerned only with the centres of the circles, hence the other group may be taken as points. If R is the radius of each of the infinite circles and α, β, γ the angles at which they intersect, $\Delta(A, B, C) = R(\sin\alpha + \sin\beta + \sin\gamma)$ and $(Dd) = -\rho^2$, where ρ is the infinite radius of the circle orthogonal to A, B, and C, see §5. Now ρ^2 is the power with respect to either of the circles A, B or C of the radical centre, which becomes the centre of the inscribed circle of the triangle formed by A, B and C; this is readily shown to be $2Rr$, r being the radius of the inscribed circle. Substituting and rejecting the infinite factor R^2, we have, on dividing by 8,

$$\begin{vmatrix} p_1 & p_1' & p_1'' \\ p_2 & p_2' & p_2'' \\ p_3 & p_3' & p_3'' \end{vmatrix} = \Delta(a, b, c)\, r\,(\sin\alpha + \sin\beta + \sin\gamma). \qquad (23)$$

The factor $r(\sin\alpha + \sin\beta + \sin\gamma)$ is equivalent to the area of the triangle formed by the lines A, B and C divided by the radius of its circumscribed circle. Thus the theorem is: Given three points and three straight lines, the determinant of the nine perpendiculars is equal to twice the product of the areas of the triangles formed by the points and by the lines divided by the radius of the circle circumscribing the latter. This theorem is also readily derived by the multiplication of two determinants whose values are $\Delta(a, b, c)$ and $\dfrac{\text{area } ABC}{R}$.

Finally, if both groups consist of straight lines, Π vanishes since D and d are both at infinity, so that $(dD) = 0$, the elements of the determinant now being the cosines of angles such that the corresponding differences in the several rows are equal.

The Power Determinant for Pairs of Circles.

21. Now let a, b and A, B be any two pairs of circles; take for C in equation (12) any circle of the system orthogonal to a, b, and for c, any circle of the system orthogonal to A, B. Then $\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix}$ reduces to $(cC)\,\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix}$. Since D is orthogonal to a, b and c, and d to A, B and C, we may state the conditions imposed upon C, D, c and d thus: C and D are two of the system orthogonal to a and b, and c, d are two of the system orthogonal to A and B,

with the condition that c is orthogonal to D, and d to C. Equation (12) then becomes

$$\Pi\begin{pmatrix} a, & b \\ A, & B\end{pmatrix} = \begin{vmatrix} (aA) & (aB) \\ (bA) & (bB)\end{vmatrix} = -4\Delta(a, b, c)\,\Delta(A, B, C)\frac{(dD)}{(cC)}.$$

Let us for the moment denote the centres of the circles by C, D, c and d; then C and D are upon the radical axis of a, b, and c and d upon that of A, B. Let P_c denote the perpendicular from c upon the line of centres of a, b, and P_C the perpendicular from C upon the line of centres of A, B; then the equation becomes

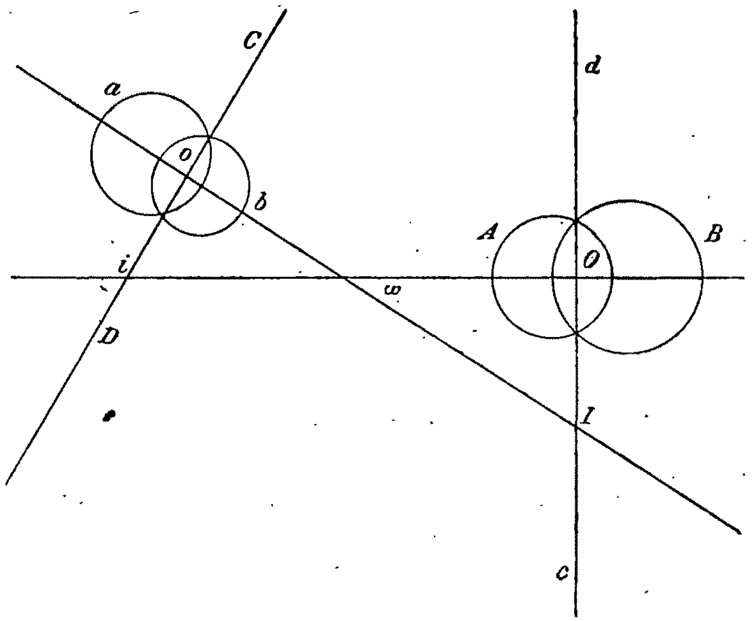
$$\Pi\begin{pmatrix} a, & b \\ A, & B\end{pmatrix} = -4\overline{ab}\,.\,\overline{AB}\,P_cP_C\frac{(dD)}{(cC)}, \qquad (24)$$

where $\overline{ab}$ and $\overline{AB}$ are the distances between the centres.

Let the common chord of a, b be $2l$ and o its middle point; let the radical axis of a, b meet the line of centres of A, B in i, and let ω be the angle between the lines of centres.

Now let D in (24) be placed at i; then the circle c becomes the line of centres of A, B, and the point c is at an infinite distance ρ, and we shall have

$$P_c = \rho\cos\omega, \quad (cC) = 2\rho P_C.$$



Substituting in (24), we have

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = \begin{vmatrix} (aA) & (aB) \\ (bA) & (bB) \end{vmatrix} = -\,2\overline{ab}\,\overline{AB}\cos\omega\,(dD). \tag{25}$$

In this equation the point d is any point upon the radical axis of A, B, but (dD) is determinate, for the centre of D is now at i on the radical axis of the system to which c and d belong. Thus

$$(dD) = \overline{iO}^2 + L^2 - (\overline{io}^2 - l^2) = \overline{iI}^2 - \overline{IO}^2 - \overline{io}^2 + L^2 + l^2, \tag{26}$$

If the circles a, b do not intersect, we must of course substitute for l^2, $-r^2$, where r is the distance from o of the limiting points of the coaxial system to which a and b belong.

22. When $\omega = 90°$, the value of Π does not vanish, because (dD) is then infinite. In this case, let the perpendicular from o to the line of centres of A, B be denoted by h, and that from O to the line of centres of a, b by H. Then the limiting value of $(dD)\cos\omega$ is readily shown to be $-\,2hH$; and we have when the lines of centres are at right angles,

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = 4\overline{ab}\,.\,\overline{AB}\,hH. \tag{27}$$

In fact, it may be shown that in this case $\dfrac{(dD)}{(cC)}$ in equation (24) becomes -1. In (27) H and h are positive when on the left of the direction a to b, and A to B respectively.

23. When $\overline{ab} = 0$; that is, when the circles a and b are concentric, Π again assumes an indeterminate form. Since ω is now indeterminate, we may employ (27), in which h is now infinite. From the properties of the radical axis we have $\overline{ao}^2 - \overline{bo}^2 = r_1^2 - r_2^2$, whence at the limit $\overline{ab}\,.\,2h = r_1^2 - r_2^2$ and the equation becomes

$$\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix} = 2\,(r_1^2 - r_2^2)\,\overline{AB}\,.\,H. \tag{28}$$

In like manner, when both pairs of circles are concentric, the expression reduces, as it should, to $(r_1^2 - r_2^2)(R_1^2 - R_2^2)$.

24. The value of $\Pi\begin{pmatrix} a, & b \\ A, & B \end{pmatrix}$ vanishes only when (dD) defined by (26) van-

ishes; in other words, when one of the system of circles orthogonal to one pair of circles, say a, b, belongs also to the coaxial system defined by the other pair A, B.

Application to Spheres.

25. The relative power of two spheres is defined in the same manner as that of two circles, so that it is the same as that of the sections made by any plane passing through the centres. It is also the same for the sections made by any plane passing through one of the centres, but not for any plane whatever.

The application to spheres of the methods employed above is so obvious that we shall only mention the general results, which of course require similar modification in the special cases when one or more of the spheres is replaced by a plane or by the sphere at infinity.

Thus, we have for any group of five spheres the determinant $\Delta(a, b, c, d, e)$ whose rows are of the form

$$x^2 + y^2 + z^2 - r^2, \quad x, \quad y, \quad z, \quad 1,$$

(x, y, z and r being in each row the rectangular coordinates of the centre and the radius of one of the spheres), and whose value is independent of the position of the origin and axes. This implies a linear relation between the powers of any sixth sphere relatively to the five given spheres. The value of $\Delta(a, b, c, d, e)$ is the power of one of the spheres relatively to that which cuts the other four orthogonally multiplied by six times the volume of the tetrahedron whose vertices are the centres of these four spheres,

$$\Delta(a, b, c, d, e) = 0$$

is the condition that five spheres shall have a common orthogonal sphere, and

$$\Delta(a, b, c, d, z) = \text{constant}$$

imposes upon the variable sphere z the condition that it shall cut orthogonally a certain sphere whose centre is the radical centre of a, b, c and d.

26. The power determinant of two groups of five spheres, each is

$$\Pi\begin{pmatrix} a, & b, & c, & d, & e \\ A, & B, & C, & D, & E \end{pmatrix} = 8\Delta(a, b, c, d, e)\,\Delta(A, B, C, D, E), \tag{29}$$

which depends only upon the configurations of the two groups, and not upon their relative positions, and which vanishes when either group has a common orthogonal sphere.

The Power Determinant for Groups of Four Spheres.

27. For the power determinant of two groups of four spheres each we derive as in §16,

$$\left.\begin{aligned}\Pi\begin{pmatrix} a, & b, & c, & d \\ A, & B, & C, & D\end{pmatrix} &= \begin{vmatrix} (aA) & (aB) & (aC) & (aD) \\ (bA) & (bB) & (bC) & (bD) \\ (cA) & (cB) & (cC) & (cD) \\ (dA) & (dB) & (dC) & (dD)\end{vmatrix} \\ &= 8\Delta(a,\, b,\, c,\, d)\,\Delta(A,\, B,\, C,\, D).(eE)\end{aligned}\right\} \quad (30)$$

where $\Delta(a,\, b,\, c,\, d)$ now denotes six times the volume of the tetrahedron whose vertices are the centres of a, b, c and d, and e and E are the spheres orthogonal respectively to the two groups.

That is, the power determinant of two groups of four spheres each is 288 *times the product of the volumes of the tetrahedra formed by the centres of the groups and the relative power of the spheres respectively orthogonal to the two groups.*

It follows that this determinant *vanishes when the spheres respectively orthogonal to the two groups cut each other at right angles.*

When the two groups are identical we have

$$\Pi\begin{pmatrix} a, & b, & c, & d \\ a, & b, & c, & d\end{pmatrix} = -\,16\Delta^2(a,\, b,\, c,\, d)\,R^2, \qquad (31)$$

R being the radius of the sphere orthogonal to the group.

The Power Determinant for Groups of Three Spheres.

28. For groups of three spheres we obtain as in §21,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C\end{pmatrix} = 8\Delta(a,\, b,\, c,\, d)\,\Delta(A,\, B,\, C,\, D)\,\frac{(eE)}{(dD)},$$

where d and e are orthogonal to A, B, C, and D, E to a, b, c; D being also orthogonal to e and E to d. Operating as before, this becomes

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C\end{pmatrix} = 4\Delta(a,\, b,\, c)\,\Delta(A,\, B,\, C)\cos\omega\,(eE), \qquad (32)$$

in which $\Delta(a,\, b,\, c)$ and $\Delta(A,\, B,\, C)$ are the double areas of the triangles whose vertices are the centres of the spheres of the respective groups, ω is the angle between the planes of centres, and

$$(eE) = \overline{iI}^2 - \overline{io}^2 - \overline{IO}^2 + l^2 + L^2, \qquad (33)$$

where the two groups of spheres are supposed to have the common chords $2l$ and $2L$, o and O being their middle points, and where i is the point in which the radical axis of a, b, c meets the plane of centres of A, B, C, I the point in which the radical axis of A, B, C meets the plane of centres of a, b, c.

In equation (32), ω is the angle between those aspects of the planes in which $\Delta(A, B, C)$ and $\Delta(a, b, c)$ have the same sign. When the planes are coincident, $\omega = 180°$ and the equation reduces to equation (12), Io and iO vanishing in the value of (eE).

29. If the three spheres a, b, c do not intersect, we must replace l^2 by $-r^2$, where r is the radius of the circle orthogonal to the three spheres. When this circle is real for each group we have

$$(eE) = \overline{iI}^2 - \overline{io}^2 - \overline{IO}^2 - r^2 - R^2,$$

and this quantity may be defined as the relative power of the two plane circles in question, which are not in the same plane, whose centres are at o, O and whose radii are r, R; i and I being points in each of which the axis of one circle cuts the plane of the other.

From the mode in which (eE) arises, it is obviously the same as the relative power of two spheres passing through the circles and having both their centres in the plane of one of the circles. It vanishes when the circles are such that a sphere can be passed through one of them which shall cut the other circle orthogonally.

The power determinant of two groups of three spheres each vanishes when the circles respectively orthogonal to the groups have this relation.

30. When the planes of the centres of the two groups are perpendicular, the value of Π, when reduced as in §22, becomes

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = -8\Delta(a, b, c)\Delta(A, B, C)hH, \tag{34}$$

where h and H are the distances of o and O from the planes of the centres of A, B, C and a, b, c respectively.

31. When $\Delta(a, b, c) = 0$, that is, when the centres of a, b and c are in a straight line, we may take the plane of centres perpendicular to that of A, B, C, and so employ (34), in which h is now infinite. In fact, if R is the infinite radius of the circle orthogonal to a, b, c, and ϕ the inclination of the line of

centres to the plane of centres of A, B, C, we have $h = R\cos\phi$; hence, employing the notation of §§17 and 18, we have by equation (15), disregarding sign,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 4\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)\cos\phi\,\Delta\,(A,\,B,\,C)\,H. \qquad (35)$$

When $\phi = 0$, this reduces to equation (16).

In like manner, if the centres of A, B, C are also in a straight line,

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix}$$
$$= 2\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)(D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2)\cos\phi\cos\psi,$$

where ϕ and ψ are the inclinations of the lines of centres of a, b, c and A, B, C to two mutually perpendicular planes passed through the line of centres of A, B, C and a, b, c respectively. Thus, if θ is the angle between two lines parallel to the lines of centres, we have in this case

$$\Pi\begin{pmatrix} a, & b, & c \\ A, & B, & C \end{pmatrix} = 2\,(d_1d_2d_3 + d_1r_1^2 + d_2r_2^2 + d_3r_3^2)(D_1D_2D_3 + D_1R_1^2 + D_2R_2^2 + D_3R_3^2)\cos\theta. \qquad (36)$$

This is identical with equation (17), in which, however, the two lines of centres were supposed to intersect.

Application of Quaternions to Projective Geometry.

By C. H. Chapman.

Introduction.

In the following article I employ Hamilton's complex numbers[*] of the type $xi + yj + zk$. The real or complex quantities x, y, z are interpreted as the trilinear coordinates of a point in the plane of the triangle of reference. The properties of i, j and k[†] are utilized, but their possible and usual geometrical interpretation as directed lines in space of three dimensions is, for the present, left aside. The number $\rho = xi + yj + zk$ is spoken of as the affix of the point of which x, y, z are the trilinear coordinates; or, dualistically, $\alpha = ai + bj + ck$ is the affix of the line of which a, b, c are the coordinates.

The point equation of a line is $S\alpha\rho = 0$; here α is the affix of the line and ρ varies; but if ρ is constant and α varies, we obtain the pencil of lines through the point ρ. In that case $S\alpha\rho = 0$ is the equation of the point ρ. The equation of a line through the points α and β is

$$S.\rho V\alpha\beta = 0 \text{ or } S.\rho\alpha\beta = 0;$$

but this may also be interpreted as the equation of the point of intersection of the lines whose affixes are α and β.

The homogeneous equation of the second degree in three variables takes the form

$$S\rho\phi\rho = 0,$$

ϕ being the self-conjugate linear and vector function whose matrix is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

---

[*] Hankel, Complexe Zahlensysteme, p. 141.

[†] Tait's Quaternions, p. 36.

Starting from this, I show that the tangential equation of the conics is

$$S\alpha\phi^{-1}\alpha = 0,$$

which, for a pair of lines, becomes simply

$$(S.\sigma\tau_1)^2 = 0,$$

τ_1 being the affix of their point of intersection.

The relation of pole and polar is expressed by ϕ; if α is the affix of the pole, then $\phi\alpha$ is the affix of the polar. Conversely, given the polar, we pass to the pole by operating with ϕ^{-1}.

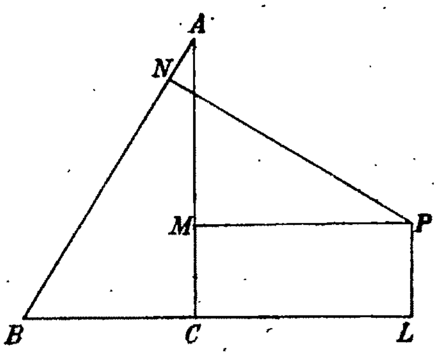
In case the matrix has one latent root equal to zero, the conic breaks up and ϕ^{-1} is indeterminate while ϕ is not. We can therefore pass from the pole to the polar, but not from polar to pole when the conic is a pair of lines.

In this new field, quite as much as in the more familiar ones, the calculus of quaternions shows its power by going directly to the result without analytical artifice, and by producing very simple formulas whose interpretation is intuitive. Especial attention might be called to the fact that the reciprocal polar relation is emphasized by the appearance of ϕ^{-1} in the tangential equation.

In Part 4 the method is extended to plane cubics, and there is nothing to hinder its application to curves in general, as I hope to show in a future paper.

Constant references are made to Tait's Quaternions for formulas, sometimes perhaps of a too elementary character to need reference. The second edition is used.

1.—*Equation of the Right Line.*



Consider a triangle ABC and a point P in its plane. From P let fall the perpendiculars PL, PM, PN on the sides of the triangle. There is no other point in the plane from which all the perpendiculars on the sides of the triangle will have the same lengths in the same order as those from P. Three perpendiculars chosen and arranged arbitrarily will not in general attach themselves in this way to any point, for the ends of any pair of them may be brought together, but the third one will then be found too short or too long. In fact the third is known when any two are given by virtue of the relation

$$p_1 a + p_2 b + p_3 c = 2K, \tag{1}$$

p_1, p_2, p_3 being the perpendiculars on the sides a, b, c, and K being the area of the triangle.

Each point in the plane has one set of perpendiculars attached to it, and no two points have the same set; but put

$$p_1 = lp_1', \; p_2 = mp_2', \; p_3 = np_3', \; n,$$

where l, m, n are any constants whatever, and make

$$al = A, \; bm = B, \; cn = C,$$

the relation (1) becomes

$$p_1'A + p_2'B + p_3'C = 2K,$$

and we conclude that there is a set of three numbers, proportional to arbitrary multiples of the perpendiculars on the sides of the triangle, attached to each point, and that no two points have the same set.

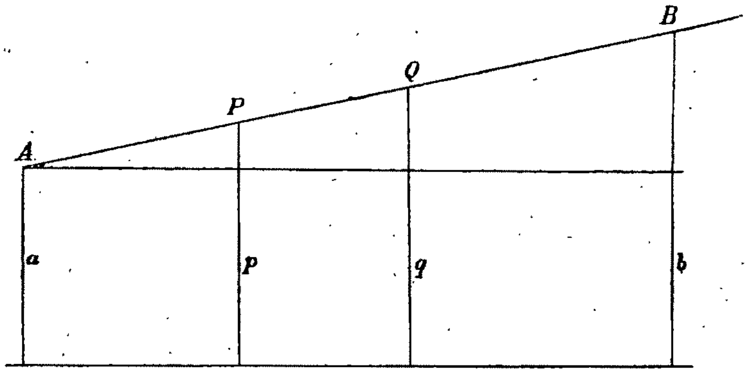
Let Hamilton's units, i, j, k, be attached as marks of distinction to distances measured along PL, PM, PN respectively; and let the perpendiculars from P, or fixed arbitrary multiples of them, be denoted by x, y, z; then the complex number $xi + yj + zk$ will be associated with the point P in such a way that when P is known the number is known, and conversely. The number $xi + yj + zk$ will be called the affix of the point P.

Numbers of this kind enjoy all the mathematical properties of vectors,* only the formulas obtained with them will now bear a new geometrical interpretation. They will be denoted, as is customary with vectors, by Greek letters.

---

* Hankel, Complexe Zahlensysteme, §42.

If three points lie on a line, their affixes are connected by a linear homogeneous relation.



Let P and Q be the points lying on the line AB, and let the i-components of the affixes of A, P, Q be a, p, q respectively. We have from the figure $\frac{p-a}{q-a}=\frac{AP}{AQ}=\frac{\lambda}{\mu}$, say; whence $\mu p-\lambda q+(\lambda-\mu)a=0$. The same relation holds for the j and k components of the affixes of A, P, Q and hence for the affixes themselves. Conversely, if the affixes of three points are connected by a linear homogeneous relation, the points lie on a line.

Let $\alpha+m\beta+n\gamma=0$ be the relations between the affixes α, β, γ; we have only to make $\frac{\mu}{\lambda-\mu}=m$, $\frac{\lambda}{\mu-\lambda}=n$; thus a point is determined on the line through α and β whose affix is $\gamma=-\frac{\alpha+m\beta}{n}$.

Operating with $V.\beta\gamma$ and taking scalars, we deduce from the relation

$$\alpha+m\beta+n\gamma=0$$

the quaternion form
$$S.\alpha\beta\gamma=0, \tag{1}$$

which is therefore the necessary and sufficient condition that α, β, γ lie on a line.

Since the relation

$$(\alpha+\lambda\beta)-\alpha-\lambda\beta=0$$

is satisfied whatever λ may be, we conclude that the point $\alpha+\lambda\beta$ always lies on the line through α and β; and as λ varies through all real values, it represents successively all the points on the line.

The points α, β, γ lie on a line so long as (1) is satisfied; hence if

$$S.\rho V\alpha\beta=0=S.\rho\alpha\beta,$$

the locus of ρ is the straight line through α and β. Similarly, if $V\alpha\beta$ varies subject to (1), we shall have the equation of the straight line in the form

$$S.a\rho = 0. \tag{2}$$

This line is the locus of all points γ given by the equation

$$\gamma = xV\sigma\alpha, \tag{3}$$

where x and σ may have any values, x being a scalar. We conclude from (2) that $V.\alpha\beta$ is the affix of the intersection of the lines $S.a\rho = 0$, $S.\beta\rho = 0$.

The equations of the sides of the triangle of reference are

$$Si\rho = 0, \; Sj\rho = 0, \; Sk\rho = 0.$$

These equations are found by noting that the affix of the vertex B is pj; of C, qk; hence the equation of BC is $S.\rho Vjk = 0$; but $Vjk = jk = i$.

2.—*Transformation of Affixes.*

Let ρ be the affix of a point referred to a certain triangle; its affix referred to a triangle having $\delta_1, \delta_2, \delta_3$ for the affixes of its vertices is required. We have

$$\rho S\delta_1\delta_2\delta_3 = \delta_1 S\rho\delta_2\delta_3 + \delta_2 S\rho\delta_1\delta_3 + \delta_3 S\rho\delta_2\delta_1.$$

Now $S.\rho\delta_2\delta_3$, ρ being any point, is proportional to the perpendicular from the point ρ on the line through δ_2 and δ_3. We may therefore write

$$\rho = X\delta_1 + Y\delta_2 + Z\delta_3,$$

where X, Y, Z are, for all points, proportional to the perpendiculars on the sides of the new triangle. Further than this, if $\delta_1 = ai + bj + ck$, then

$$\left[\frac{\delta_1}{\sqrt{a^2 + b^2 + c^2}}\right]^2 = -1;$$

so that, by including $\sqrt{a^2 + b^2 + c^2}$ and the corresponding factors for δ_2 and δ_3 in X, Y, Z, we may write

$$\rho = X\alpha + Y\beta + Z\gamma,$$

where $\alpha^2 = \beta^2 = \gamma^2 = -1$; although in general the system α, β, γ has not the properties of i, j, k. We shall have $\alpha\beta = -\beta\alpha$ if $S.\alpha\beta = 0$, and not otherwise; in case then $\alpha^2 = \beta^2 = \gamma^2 = -1$, and $\alpha\beta = -\beta\alpha$, $\alpha\gamma = -\gamma\alpha$, $\beta\gamma = -\gamma\beta$, we shall also have $\alpha\beta = \gamma$, $\beta\gamma = \alpha$, $\gamma\alpha = \beta$. The proof is just the same as it would be if α, β, γ represented a set of rectangular unit vectors. Such a system of affixes may be called orthogonal. The affixes determined by the equation

$$(\phi - g)\rho = 0$$

form an orthogonal system if $\phi\rho$ is a self-conjugate* function linear in ρ.

We may also effect a linear transformation by writing $\rho = \psi\rho'$, where ψ is a matrix of order three.

3.—*Projective Geometry of Conics.*

The homogeneous equation of the conics

$$ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx = 0 \qquad (1)$$

may be written in the form

$$S.\rho\phi\rho = 0, \qquad (2)$$

where $\rho = xi + yj + zk$ and $\phi\rho = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}(xi + yj + zk)$.

Let τ and σ be the affixes of two points; any point on the line joining them has for its affix $\tau + \lambda\sigma$, λ being a scalar. Substituting this for ρ in (2) we shall obtain for λ the quadratic

$$S(\tau + \lambda\sigma)\,\phi\,(\tau + \lambda\sigma) = 0,$$

or

$$S.\tau\phi\tau + \lambda S.(\sigma\phi\tau + \tau\phi\sigma) + \lambda^2 S.\sigma\phi\sigma = 0.$$

But since ϕ is self-conjugate, this may be written

$$\lambda^2 S.\sigma\phi\sigma + 2\lambda S.\sigma\phi\tau + S.\tau\phi\tau = 0. \qquad (3)$$

If the point τ be taken fixed and σ variable, then

$$S.\sigma\phi\tau = 0 \qquad (4)$$

is the equation of a line, the polar line of τ with respect to the conic. This equation is symmetrical in σ and τ, as it should be.

Let $S.\sigma\alpha = 0$ be the equation of a polar line; then $\phi^{-1}\alpha$ is the affix of its pole. Now we know that

$$m\phi^{-1}\alpha = (m_1 - m_2\phi + \phi^2)\,\alpha;\dagger \qquad (5)$$

hence $\phi^{-1}\alpha$ can always be determined unless $m = 0$; but m is the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

---

* Tait, p. 90.

† Tait, p. 82.

In case this determinant vanishes, the pole of the given line cannot be determined.

If the point τ lies on its own polar, it must satisfy equation (4), which becomes $S.\tau\phi\tau = 0$; hence τ also satisfies equation (2) and lies on the curve.

We had above $\tau = \phi^{-1}\alpha$; hence $S.\tau\phi\tau = S.\alpha\phi^{-1}\alpha = 0$. The condition

$$S.\alpha\phi^{-1}\alpha = 0 \tag{6}$$

being satisfied by the affixes of lines tangent to the conic, is the tangential equation of the conic.

We see that if τ is the affix of the pole, then $\phi\tau$ is the affix of the polar; and when $m = 0$, we can determine τ so that it shall satisfy the relation

$$\phi\tau_1 = 0. \tag{7}$$

This requires a nullity of order 1 in the matrix ϕ; in other words, one latent root of ϕ is zero.*

In this case it makes no difference what line in the plane is the locus of σ, we shall always have

$$S.\sigma\phi\tau_1 = 0;$$

so that any line may be taken for the polar of this exceptional point τ_1.

Let ω be any other point; its polar is given by

$$S.\omega\phi\rho = 0,$$

but this is satisfied when $\rho = \tau_1$, hence the polar lines of all points pass through τ_1.

We may write the equation of the polar of ω in the form

$$S.\rho\phi\omega = 0, \tag{8}$$

which is equally satisfied if we replace ω by $\omega + x\tau_1$. From this we conclude that any point on the line joining ω and τ_1 may be taken as the pole of the line (8).

The hypothesis still being that $m = 0$, let α be a point on the curve; any point on the line through τ_1 and α is $\tau_1 + x\alpha$. Let this be substituted in $S.\rho\phi\rho = 0$. We obtain

$$S.(\tau_1 + x\alpha)\,\phi\,(\tau_1 + x\alpha) = S.\tau_1\phi\tau_1 + 2xS.\alpha\phi\tau_1 + x^2S.\alpha\phi\alpha;$$

an expression in which every term vanishes. Hence the line joining τ_1 to any point on the curve forms part of the curve.

---

* Taber, *Am. Journ. Math.*, Vol. XII, p. 862.

To determine the vector τ_1 which satisfies equation (7), we recall that $m = -S.\phi i\phi j\phi k$, if i, j, k be taken for a fundamental system.* Hence in this case

$$S.\phi i\phi j\phi k = S.i\phi V.\phi j\phi k = 0.$$

We have in all cases

$$S.j\phi V.\phi j\phi k = 0,$$
$$S.k\phi V.\phi j\phi k = 0.$$

Hence for any vector whatever ρ, we have

$$S.\rho\phi V.\phi j\phi k = 0,$$

from which we must conclude that

$$\phi V.\phi j\phi k = 0,$$

and finally that

$$\tau_1 = xV.\phi j\phi k. \qquad (10)$$

We might show just as well that $\tau = xV.\phi k\phi i$, or $xV.\phi i\phi j$; hence, since the scalar equations for τ_1 are linear in the variables and it can therefore have only one value, we conclude that $V.\phi j\phi k = t_1 V.\phi k\phi i = t_2 V.\phi i\phi j$.

Let us now expand $V.\phi j\phi k$ in terms of i, j, k; assume

$$xV.\phi j\phi k = i + yj + zk,$$

a relation which must hold for some system of values of x, y, z. Operating with ϕ, we obtain the relation

$$\phi i + y\phi j + z\phi k = 0,$$

whence

$$yV.\phi j\phi k = -V.\phi i\phi k, \text{ or } y = \frac{1}{t_1}$$

and

$$zV.\phi k\phi j = -V.\phi i\phi j, \text{ or } z = \frac{1}{t_2}.$$

Therefore

$$xV.\phi j\phi k = i + \frac{1}{t_1}j + \frac{1}{t_2}k = x\tau_1. \qquad (11)$$

In general, we may write for any vector α,

$$S.\phi i\phi j\phi k.\alpha = \phi iS.\alpha\phi j\phi k + \phi jS.\alpha\phi k\phi i + \phi kS.\alpha\phi i\phi j.$$

But

$$S.\phi i\phi j\phi k = -m,$$

hence, operating with ϕ^{-1},

$$-m\phi^{-1}\alpha = iS.\alpha\phi j\phi k + jS.\alpha\phi k\phi i + kS.\alpha\phi i\phi j.$$

---

* Tait, p. 81.

From this we find

$$-mS.\alpha\phi^{-1}\alpha = S.\alpha iS.\alpha\phi j\phi k + S.\alpha jS.\alpha\phi k\phi i + S.\alpha kS.\alpha\phi i\phi j.$$

Now if $m = 0$, we may replace $V\phi j\phi k$ by τ_1 and we obtain the relation

$$S.\alpha\tau_1\left(S.\alpha i + \frac{1}{t_1}S.\alpha j + \frac{1}{t_2}S.\alpha k\right) = 0,$$

or

$$S.\alpha\tau_1 S.\alpha\left(i + \frac{1}{t_1}j + \frac{1}{t_2}k\right) = 0,$$

or finally $(S.\alpha\tau_1)^2 = 0$, by aid of (11).

This is the form assumed by the tangential equation when the conic breaks up into a pair of lines. It is the equation of the point τ_1 squared.

No line can be tangent to this conic unless it passes through τ_1; now the affix of any line may be written in the form

$$a\delta_1 + b\delta_2 + c\tau_1, \tag{12}$$

where δ_1 and δ_2 are the other two affixes determined by the equation $(\phi - g)\rho = 0$. These affixes form an orthogonal system.* Substituting (12) in the equation of the point τ_1, we obtain

$$S.\tau_1(a\delta_1 + b\delta_2 + c\tau_1) = 0,$$

or

$$c\tau_1^2 = 0,$$

whence, unless $\tau_1^2 = 0$, we conclude that $c = 0$. Any line therefore will be tangent to the pair of lines if its affix is a linear function of δ_1 and δ_2 only.

The curve

$$S.\rho\phi\rho - pS^2.\rho\phi\alpha = 0$$

is a conic touching the conic $S.\rho\phi\rho = 0$. It passes through the point α, if $p = \frac{1}{S.\alpha\phi\alpha}$, and its equation then becomes

$$S.\alpha\phi\alpha S\rho\phi\rho - S^2.\rho\phi\alpha = 0. \tag{13}$$

The points of contact are on the line $S.\rho\phi\alpha = 0$, which is the polar of α. Now we may write equation (13) in the form $S.\rho\,[\phi\rho S.\alpha\phi\alpha - \phi\alpha S.\rho\phi\alpha] = 0$; where the expression in the brackets is on its face a self-conjugate linear function of ρ. To compute for it the quantity m, we may choose α, $\phi\alpha$ and $V.\alpha\phi\alpha$ for a system of independent affixes; then designating the function by ψ, we shall have

---

* Tait, p. 89.

$$m=\frac{S.\psi\alpha\psi\phi\alpha\psi V.\alpha\phi\alpha}{S.\alpha\phi\alpha\, V.\alpha\phi\alpha};$$

but $\psi\alpha=\phi\alpha S.\alpha\phi\alpha-\phi\alpha S.\alpha\phi\alpha=0$; hence for ψ the quantity m vanishes. We conclude that the conic represented by (13) is a pair of lines, the tangents from α to the conic.

The polar lines of the points whose affixes are α, β, γ are $S.\sigma\phi\alpha=0$, $S.\sigma\phi\beta=0$, $S.\sigma\phi\gamma=0$; and if each point lies on the polars of the other two, we have in addition

$$S.\beta\phi\alpha=S.\beta\phi\gamma=0;\ S.\alpha\phi\beta=S.\alpha\phi\gamma=0;\ S.\gamma\phi\alpha=S.\gamma\phi\beta=0. \tag{14}$$

The points α, β, γ are now the vertices of a polar triangle, and if we write $\rho=\frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}}+\frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}}+\frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}$, then the equation of the conic $S.\rho\phi\rho=0$ takes the form

$$S.\left(\frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}}+\frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}}+\frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}\right)\phi\left(\frac{x\alpha}{(S.\alpha\phi\alpha)^{\frac{1}{2}}}+\frac{y\beta}{(S.\beta\phi\beta)^{\frac{1}{2}}}+\frac{z\gamma}{(S.\gamma\phi\gamma)^{\frac{1}{2}}}\right)=0,$$

or

$$x^2+y^2+z^2=0. \tag{15}$$

Let $S.\rho\phi\rho=0$ and $S.\rho\psi\rho=0$ represent two conics. The polar of a point α referred to each respectively will be given by $S.\sigma\phi\alpha=0$, $S.\sigma\psi\alpha=0$. These equations will represent the same line if, and only if,

$$\phi\alpha=x\psi\alpha;\ \text{whence}\ \alpha=x\phi^{-1}\psi\alpha,$$

or $\alpha=\frac{1}{x}\psi^{-1}\phi\alpha$, since if α satisfies one of these equations it must also satisfy the other. There are three points, then, which have the same polar lines with respect to the two conics. Their affixes are given as the roots of

$$(\phi^{-1}\psi-x)\,\alpha=0,$$

and they form a polar triangle common to the two conics. For let δ_1, δ_2, δ_3 be the roots of $(\phi^{-1}\psi-x)\,\alpha=0$, and h_1, h_2, h_3, supposed all different, the corresponding scalars; so that $\phi^{-1}\psi\delta_1=h_1\delta_1$, $\phi^{-1}\psi\delta_2=h_2\delta_2$; noting that $\psi\delta_1=h_1\phi\delta_1$, we have now $S.\phi^{-1}\psi\delta_2\psi\delta_1=h_1h_2S.\delta_2\phi\delta_1$. But $S.\phi^{-1}\psi\delta_2\psi\delta_1=h_2S.\delta_2\psi\delta_1=h_2S.\delta_1\psi\delta_2$ $=h_2^2S.\delta_1\phi\delta_2$, since $\psi\delta_2=h_2\phi\delta_2$. We have then $h_1h_2S.\delta_2\phi\delta_1=h_2^2S.\delta_2\phi\delta_1$, whence we conclude that since h_1 and h_2 are different, $S.\delta_2\phi\delta_1$ must vanish. Hence δ_2 is on the polar of δ_1. The same is true of δ_3. In this way the triangle is seen to be a polar triangle. If the equation $S.\rho\phi\rho = 0$ referred to this triangle becomes

$$x^2 + y^2 + z^2 = 0,$$

then $S.\rho\psi\rho = 0$ becomes

$$x^2 S.\alpha\psi\alpha + y^2 S.\beta\psi\beta + z^2 S.\gamma\psi\gamma = 0. \tag{16}$$

Let α and β be the affixes of any two lines; a line through their intersection has the affix $\alpha + \lambda\beta$. Substituting this in the tangential equation of the conic $S\alpha\phi^{-1}\alpha = 0$, we obtain

$$S.(\alpha + \lambda\beta)\phi^{-1}(\alpha + \lambda\beta) = 0;$$

that is,

$$S.\alpha\phi^{-1}\alpha + 2\lambda S.\beta\phi^{-1}\alpha + \lambda^2 S.\beta\phi^{-1}\beta = 0. \tag{17}$$

If λ_1, λ_2 are the roots of this equation, then

$$\alpha + \lambda_1\beta, \ \alpha + \lambda_2\beta \tag{18}$$

are the affixes of the tangents to the conic through the intersection of the lines $S.\alpha\rho = 0$, $S.\beta\rho = 0$. The lines (18) form a harmonic pencil with α and β if

$$S.\beta\phi^{-1}\alpha = 0; \tag{19}$$

that is, if β is the affix of lines passing through the pole of α. For we know that $\phi^{-1}\alpha$ is the affix of the pole when α is the affix of the polar. In fact (19) is the equation of the pole of α. If $S.\alpha\rho = 0$ passes through its own pole, we must have $S.\alpha\phi^{-1}\alpha = 0$, which means of course that α is tangent to the conic.

Let $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$ be any two conics which do not break up. The conic $S.\rho(\phi - \lambda\psi)\rho = 0$, (20), passes through their points of intersection. For the function $\phi - \lambda\psi$, which is self-conjugate, the quantity m has the value $-S.[(\phi - \lambda\psi)i(\phi - \lambda\psi)j(\phi - \lambda\psi)k]$; hence, when m vanishes, λ satisfies a cubic. There are then three values of λ for which $S.\rho(\phi - \lambda\psi)\rho = 0$ breaks up into a pair of lines. The double points on these pairs are the vertices of a polar triangle common to all conics of the system (20). For, if

$$S.\rho(\phi - \lambda_1\psi)\rho = 0 \text{ and } S.\rho(\phi - \lambda_2\psi)\rho = 0$$

are any two conics of the system, the affix α of a vertex of their common polar triangle is given by the equation $(\phi - \lambda_1\psi)\alpha = x(\phi - \lambda_2\psi)\alpha$, which is possible in general only if $\phi\alpha = g\psi\alpha$; that is, the polar triangle common to any two conics of the system is the one common to $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$. But the equation may also subsist if $(\phi - \lambda_1\psi)\alpha = 0$, $(\phi - \lambda_2\psi)\alpha = 0$; that is, if each of the two conics breaks up into a pair of lines which intersect at α. Hence the three values of λ for which $S.\rho(\phi - \lambda\psi)\rho = 0$ breaks up, are the values of x for which $\phi\alpha = x\psi\alpha$; and the double points on the lines are the vertices of the polar triangle common to the whole system. Hence these double points have the affixes δ_1, δ_2, δ_3 which satisfy the equation

$$\phi\alpha = x\psi\alpha,$$

and are such that $\psi\delta_1 = h_1\phi\delta_1$, $\psi\delta_2 = h_2\phi\delta_2$, $\psi\delta_3 = h_3\phi\delta_3$, and the three values of λ are $\frac{1}{h_1}$, $\frac{1}{h_2}$, $\frac{1}{h_3}$ respectively.

When the conics are referred to δ_1, δ_2, δ_3 as a triangle of reference, their equations become

$$\left.\begin{array}{l} x^2 S.\delta_1\phi\delta_1 + y^2 S.\delta_2\phi\delta_2 + z^2 S.\delta_3\phi\delta_3 = 0, \\ x^2 S.\delta_1\psi\delta_1 + y^2 S.\delta_2\psi\delta_2 + z^2 S.\delta_3\psi\delta_3 = 0. \end{array}\right\} \tag{21}$$

Remembering that $\psi\delta_1 = h_1\phi\delta_1$, etc., the second of equations (21) becomes

$$h_1 x^2 S.\delta_1\phi\delta_1 + h_2 y^2 S.\delta_2\phi\delta_2 + h_3 z^2 S.\delta_3\phi\delta_3 = 0.$$

Now, including $S.\delta_1\phi\delta_1$, $S.\delta_2\phi\delta_2$, $S.\delta_3\phi\delta_3$ in x^2, y^2, z^2 respectively, their final form is

$$\begin{array}{l} x^2 + y^2 + z^2 = 0, \\ h_1x^2 + h_2y^2 + h_3z^2 = 0. \end{array}$$

These equations may be easily solved for the intersections of the two conics.

There is an interesting correspondence, which this treatment brings into prominence, between polar triangles in a conic, and systems of conjugate diameters in central surfaces of the second order. If the equation of two surfaces be $S.\rho\phi\rho = 1$ and $S.\rho\psi\rho = 1$, a system of diameters conjugate in both is given as the roots of $\phi^{-1}\psi\rho = x\rho$, which is also the equation that determines the vertices of the polar triangle common to the conics $S.\rho\phi\rho = 0$ and $S.\rho\psi\rho = 0$.*

4.—*Plane Cubics.*

If we write

$$\begin{aligned} -\varepsilon &= a_1 i + b_1 j + c_1 k \\ -\eta &= b_1 i + b_2 j + c_2 k \\ -\gamma &= c_1 i + c_2 j + c_3 k \\ -\beta &= b_2 i + b_3 j + c_4 k \\ -\delta &= c_2 i + c_4 j + c_5 k \\ -\varkappa &= c_3 i + c_5 j + c_6 k \end{aligned}$$

---

* Tait's Quaternions, Art. 269.

and
$$\rho = pi + qj + rk,$$
then the matrix
$$\begin{pmatrix} S.\varepsilon\rho & S.\eta\rho & S.\gamma\rho \\ S.\eta\rho & S.\beta\rho & S.\delta\rho \\ S.\gamma\rho & S.\delta\rho & S.\varkappa\rho \end{pmatrix},$$

which may be denoted by ϕ_ρ, has the following properties:

1). $\phi_{\rho+\sigma} = \phi_\rho + \phi_\sigma$.
2). $S.\alpha\phi_\beta\beta = S.\beta\phi_\beta\alpha = S.\alpha\phi_\alpha\beta$.
3). $S.\gamma\phi_\alpha\beta = S.\beta\phi_\alpha\gamma = S.\alpha\phi_\beta\gamma = S.\alpha\phi_\gamma\beta$.
4). $\phi_\alpha\beta = \phi_\beta\alpha$.
5). $dS.\rho\phi_\rho\rho = 3S.d\rho\phi_\rho\rho$.

The equation
$$S.\rho\phi_\rho\rho = 0 \tag{1}$$
represents a plane cubic in general form; and if the content of ϕ_ρ be denoted by m_ρ, then
$$m_\rho = 0 \tag{2}$$
is the equation of the Hessian of the cubic. The curves 1) and 2) intersect in nine points. Let α be one of the nine; we have
$$S.\alpha\phi_\alpha\alpha = 0 \tag{3}$$
and
$$m_\alpha = 0. \tag{4}$$
There is consequently a point τ for which
$$\phi_\alpha\tau = 0. \tag{5}$$
If τ coincides with α, so that
$$\phi_\alpha\alpha = 0, \tag{6}$$
then α is a double point on the cubic.

To show this, we observe that the points where the line joining β and δ cuts the cubic are given by writing in $\beta + \lambda\delta$ the values of λ which satisfy
$$S.(\beta + \lambda\delta)(\phi_\beta + \lambda\phi_\delta)(\beta + \lambda\delta) = 0,$$
or
$$S.\beta\phi_\beta\beta + 3\lambda S.\delta\phi_\beta\beta + 3\lambda^2 S.\delta\phi_\beta\delta + \lambda^3 S.\delta\phi_\delta\delta = 0. \tag{7}$$
If $\beta = \alpha$ and $\phi_\alpha\alpha = 0$, two roots of this equation are equal to zero, whatever point δ may be. Hence α is a double point. All three roots vanish if δ lies on the conic $S.\delta\phi_\alpha\delta = 0$. But when $m_\alpha = 0$ this conic breaks up into a pair of lines. If then δ lies on either of these lines, the line joining α to δ meets the

cubic at α in three coincident points; but since $\phi_{\alpha}\alpha = 0$, α is the double point on the conic. Hence the tangents to the cubic at α are the lines into which the conic $S.\rho\phi_{\alpha}\rho$ breaks up.

If not only $\phi_{\beta}\beta = 0$, but also $\phi_{\delta}\delta = 0$, then equation (7) vanishes identically, and we conclude that the line joining β and δ forms part of the curve. If there are three double points given by the relations

$$\phi_{\alpha}\alpha = 0, \quad \phi_{\beta}\beta = 0, \quad \phi_{\gamma}\gamma = 0,$$

we may write $\rho = x\alpha + y\beta + z\gamma$, and this value substituted in

$$S.\rho\phi_{\rho}\rho = 0$$

reduces the equation to

$$xyz = 0. \tag{8}$$

Change δ to ρ in equation (7) and we see without difficulty that the polar line of β with respect to the cubic is given by

$$S.\rho\phi_{\beta}\beta = 0 \tag{9}$$

and the polar conic by

$$S.\rho\phi_{\beta}\rho = 0, \tag{10}$$

while (9) also gives the polar of β with respect to the conic (10). If β is on the cubic, (9) gives the tangent at β. The point ρ cannot traverse the loci (9) and (10) simultaneously unless the conic breaks up into two lines, one of which is the tangent to the cubic at β. But if, β being on the cubic, equations (9) and (10) are satisfied, β is a point of inflection. It is the polar conics of points of inflection therefore which break up into right lines; hence it is for points of inflection β that ϕ_{β} has the property indicated by the equation

$$\phi_{\beta}\tau = 0$$

for some point τ.

Let β_1, β_2 be two points of inflection, and τ_1, τ_2 two numbers such that

$$\phi_{\beta_1}\tau_1 = 0, \quad \phi_{\beta_2}\tau = 0.$$

From the expression $V. V.\tau_1\beta_1 V.\tau_2\beta_2$ we derive the equation

$$\beta_1 S.\tau_1\tau_2\beta_2 - \tau_1 S.\beta_1\tau_2\beta_2 + \beta_2 S.\tau_2\tau_1\beta_1 - \tau_2 S.\beta_2\tau_1\beta_1 = 0, \tag{11}$$

which may be written

$$\beta_1 + y\tau_1 + z\beta_2 + w\tau_2 = 0, \tag{12}$$

where
$$y = -\frac{S.\beta_1\tau_2\beta_2}{S.\tau_1\tau_2\beta_2}; \quad z = -\frac{S.\tau_1\tau_2\beta_1}{S.\tau_1\tau_2\beta_2}; \quad w = -\frac{S.\tau_1\beta_1\beta_2}{S.\tau_1\tau_2\beta_2}.$$

By multiplying (12) by $\phi_{\beta_1}\beta_1$ and taking scalars, we find another value for z,

$$z = -\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2}. \tag{13}$$

It is seen immediately that the line joining any two points β_1, β_2 on the cubic cuts the curve again in the point

$$\beta_3 = \beta_1 + z\beta_2, \tag{13'}$$

z having the value (13). Taking β_1 and β_3 points of inflection, let us find the effect of ϕ_{β_3} upon $\beta_1 + w\tau_2$. We have

$$\phi_{\beta_3}(\beta_1 + w\tau_2) = \phi_{\beta_1}\beta_1 + z\phi_{\beta_1}\beta_2 + w\phi_{\beta_1}\tau_2.$$

But from (12),

$$\phi_{\beta_1}\beta_1 + z\phi_{\beta_1}\beta_2 + w\phi_{\beta_1}\tau_2 = 0.$$

Hence β_3 possesses the characteristic property of a point of inflection and is such that

$$\phi_{\beta_3}\tau_3 = 0,$$

where

$$\tau_3 = \beta_1 + w\tau_2 = -y\tau_1 - z\beta_2. \tag{14}$$

Upon the line joining β_1 and β_2 there can be no more than two points ρ such that

$$\phi_\rho\rho = p\rho, \tag{15}$$

where p is a scalar. For take $\rho = \beta_1 + t\beta_2$, then (15) becomes

$$\phi_{\beta_1}\beta_1 + 2t\phi_{\beta_1}\beta_2 + t^2\phi_{\beta_2}\beta_2 = p(\beta_1 + t\beta_2). \tag{16}$$

Multiplying both numbers by $V.\beta_1\beta_2$ and taking scalars, we find

$$S.\beta_1\beta_2\phi_{\beta_1}\beta_1 + 2tS.\beta_1\beta_2\phi_{\beta_1}\beta_2 + t^2S.\beta_1\beta_2\phi_{\beta_2}\beta_2 = 0, \tag{17}$$

a quadratic in t. There are no more than two values of t which satisfy (17).

If there are three points, ρ_1, ρ_2, ρ_3, mutually orthogonal and such that

$$\phi_{\rho_1}\rho_1 = p_1\rho_1;\quad \phi_{\rho_2}\rho_2 = p_2\rho_2;\quad \phi_{\rho_3}\rho_3 = p_3\rho_3, \tag{18}$$

then upon the line joining any two of them there lie three points of inflection.

It may be noted, to begin with, that $\phi_{\rho_1}\rho_2$ is orthogonal to both ρ_1 and ρ_2, since $S.\rho_1\phi_{\rho_1}\rho_2 = S.\rho_2\phi_{\rho_1}\rho_1 = p_1S.\rho_2\rho_1 = 0$; and

$$S.\rho_2\phi_{\rho_1}\rho_2 = S.\rho_1\phi_{\rho_2}\rho_2 = p_2S.\rho_1\rho_2 = 0.$$

It follows that

and in the same way,

$$\left.\begin{aligned} &\phi_{\rho_1}\rho_2 = f_3\rho_3; \\ &\phi_{\rho_2}\rho_3 = f_1\rho_1;\quad \phi_{\rho_3}\rho_1 = f_2\rho_2. \end{aligned}\right\} \tag{19}$$

Moreover, since

$$S.\rho_3\phi_{\rho_1}\rho_2 = S.\rho_2\phi_{\rho_3}\rho_1 = S.\rho_1\phi_{\rho_2}\rho_2,$$

we must have

$$f_3\rho_3^2 = f_2\rho_2^2 = f_1\rho_1^2. \tag{20}$$

Let a number $\tau = x\rho_1 + y\rho_2 + z\rho_3$ be formed, and let us inquire if there is a number $\lambda\rho_1 + \mu\rho_2$ such that

$$\phi_{(\lambda\rho_1 + \mu\rho_2)}\tau = 0;$$

that is, if

$$\lambda\phi_{\rho_1}\tau + \mu\phi_{\rho_2}\tau = 0. \tag{21}$$

Equation (21) gives the vector equation

$$\lambda(xp_1\rho_1 + yf_3\rho_3 + zf_2\rho_2) + \mu(xf_3\rho_3 + yp_2\rho_2 + zf_1\rho_1) = 0,$$

which yields the three scalar equations

$$\left.\begin{aligned} \lambda x p_1 + \mu z f_1 &= 0, \\ \lambda z f_2 + \mu y p_2 &= 0, \\ \lambda y f_3 + \mu x f_3 &= 0. \end{aligned}\right\} \tag{22}$$

These equations are consistent if $\frac{\mu}{\lambda}$ satisfies the cubic

$$\begin{vmatrix} \lambda p_1 & 0 & \mu f_1 \\ 0 & \mu p_2 & \lambda f_2 \\ \mu f_3 & \lambda f_3 & 0 \end{vmatrix} = 0, \tag{23}$$

or

$$-\lambda^3 p_1 f_2 f_3 = \mu^3 p_2 f_1 f_3,$$

whence

$$\frac{\mu}{\lambda} = -\sqrt[3]{\frac{p_1 f_2}{p_2 f_1}};\ -\omega\sqrt[3]{\frac{p_1 f_2}{p_2 f_1}};\ -\omega^2\sqrt[3]{\frac{p_1 f_2}{p_2 f_1}}, \tag{24}$$

where

$$\omega^3 = +1.$$

We may for brevity denote these values of $\frac{\mu}{\lambda}$ by

$$-c,\ -\omega c,\ -\omega^2 c,$$

where

$$c = \sqrt[3]{\frac{p_1 f_2}{p_2 f_1}}. \tag{25}$$

Equations (22) give now

$$\frac{x}{z} = -\frac{\mu}{\lambda}\,\frac{f_1}{p};\ \frac{y}{z} = -\frac{\lambda}{\mu}\,\frac{f_2}{p_2},$$

and we obtain for τ the three values

$$\left.\begin{aligned}\tau_1 &= +\ c\frac{f_1}{p_1}\rho_1 + \frac{1}{c}\,\frac{f_2}{p_2}\rho_2 + \rho_3,\\ \tau_2 &= +\ \omega c\frac{f_1}{p_1}\rho_1 + \frac{\omega^2}{c}\,\frac{f_2}{p_2}\rho_2 + \rho_3,\\ \tau_3 &= +\ \omega^2 c\frac{f_1}{p_1}\rho_1 + \frac{\omega}{c}\,\frac{f_2}{p_2}\rho_2 + \rho_3.\end{aligned}\right\} \tag{26}$$

The corresponding values of $\lambda\rho_1 + \mu\rho_2$ are proportional to

$$\beta_1 = \rho_1 - c\rho_2;\ \ \beta_2 = \rho_1 - \omega c\rho_2;\ \ \beta_3 = \rho_1 - \omega^2 c\rho_2. \tag{27}$$

If β_1, β_2, β_3 are on the cubic, they are points of inflection, since they have the characteristic property of points of inflection.

If β_1 is on the cubic, we must have

$$S.\beta_1\phi_{\beta_1}\beta_1 = 0.$$

This yields the equation

$$S.(\rho_1 - c\rho_2)(p_1\rho_1 - 2cf_3\rho_3 + c^2p_2\rho_2) = 0,$$

or

$$p_1\rho_1^3 - c^3p_2\rho_2^3 = 0. \tag{28}$$

Substituting the value of c from equation (25), this becomes

$$p_1\rho_1^3 - \frac{p_1f_2}{p_2f_1}p_2\rho_2^3 = 0.$$

Whence

$$f_1\rho_1^3 - f_2\rho_2^3 = 0,$$

which is known to be true by equation (20). In a similar way it may be shown that β_2 and β_3 are on the curve. Hence the theorem is proved.

Under this hypothesis we have

$$\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2} = \frac{f_1\rho_1^3 - \omega f_2\rho_2^3}{f_1\rho_1^3 - \omega^2 f_2\rho_2^3} = \frac{1-\omega}{1-\omega^2} = -\omega, \text{ by aid of (20);}$$

and the value previously found for β_3, that is,

$$\beta_3 = \beta_1 - \frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2}\beta_2$$

becomes

$$\begin{aligned}\beta_3 = \beta_1 + \omega\beta_2 = \rho_1 - c\rho_2 + \omega(\rho_1 - \omega c\rho_2) &= (1+\omega)\rho_1 - c(1+\omega^2)\rho_2\\ &= -\omega^2\rho_1 + c\omega\rho_2 = -\omega^2\rho_1 + c\omega\rho_2 = -\frac{1}{\omega}(\rho_1 - c\omega^2\rho_2).\end{aligned}$$

This value of β_3 differs only by a constant factor from that found in equation (27); the results found so far are therefore consistent.

We had in equation (12)

$$w = -\frac{S.\tau_1\beta_1\beta_2}{S.\tau_1\tau_2\beta_2},$$

which, in terms of ρ_1, ρ_2, ρ_3, becomes

$$w = -\begin{vmatrix} c\dfrac{f_1}{p_1} & \dfrac{1}{c}\dfrac{f_2}{p_2} & 1 \\ 1 & -c & 0 \\ 1 & -\omega c & 0 \end{vmatrix} \div \begin{vmatrix} c\dfrac{f_1}{p_1} & \dfrac{1}{c}\dfrac{f_2}{p_2} & 1 \\ \omega c\dfrac{f_1}{p_1} & \dfrac{\omega^2}{c}\dfrac{f_2}{p_2} & 1 \\ 1 & -\omega c & 0 \end{vmatrix} = -\frac{c^2 p_2}{f_2(1+2\omega)}. \tag{29}$$

With this value of w we have

$$\beta_1 + w\tau_2 = -\frac{f_2}{\omega(1-\omega)c^2 p_2}\tau_3, \tag{30}$$

which differs only by a constant factor from the value of τ_3 found in (26).

Arrived at this point we may now observe that taking

$$\rho_1 = x\beta_1 + y\beta_2,$$
$$\rho_2 = u\beta_1 + v\beta_2,$$

and c an undetermined scalar, it is possible to put β_1 and β_2 in the form

$$\left.\begin{aligned} \beta_1 &= \frac{v\rho_1 - y\rho_2}{xv - uy} = \rho_1 - c\rho_2, \\ \beta_2 &= \frac{x\rho_2 - u\rho_1}{xv - uy} = \rho_1 - \omega c\rho_2. \end{aligned}\right\} \tag{31}$$

For, from the second and third members of equations (31), we find as a possible system of values,

$$\left.\begin{aligned} x &= -\frac{\omega}{1-\omega}, & y &= \frac{1}{1-\omega}, \\ u &= \frac{-1}{c(1-\omega)}, & v &= \frac{1}{c(1-\omega)}. \end{aligned}\right\} \tag{32}$$

The arbitrary constant c may be so chosen that

$$\frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\phi_{\beta_2}\beta_2} = -\omega, \tag{33}$$

and then from equation (13′) we find

$$\beta_3 = \beta_1 + \omega\beta_2, \tag{34}$$

which, by virtue of (31), becomes

$$\beta_3 = -\frac{1}{\omega}(\rho_1 - c\omega^2\rho_2), \tag{35}$$

which differs from the value of β_3 in (27) only by the factor $\omega - 1$.

The condition $$S.\rho_1\rho_2 = 0,$$
which takes the form

$$\beta_1^2 + \omega^2\beta_2^2 + \omega S.\beta_1\beta_2 = 0, \tag{36}$$

and does not contain c explicitly, may be satisfied if $\frac{T\beta_1}{T\beta_2}$ be properly chosen.

Let ρ_3 be a number satisfying the conditions

$$S.\rho_1\rho_3 = 0, \quad S.\rho_2\rho_3 = 0, \tag{37}$$

then ρ_1, ρ_2, ρ_3 form an orthogonal system.

We next notice that owing to the conditions

$$S.\tau_1\phi_{\beta_1}\beta_1 = 0, \quad S.\beta_1\phi_{\beta_1}\beta_1 = 0, \tag{38}$$

we may write

$$\phi_{\beta_1}\beta_1 = g_1 V.\tau_1\beta_1,$$

and similarly,

$$\left.\begin{aligned}\phi_{\beta_2}\beta_2 &= g_2 V.\tau_2\beta_2, \\ \phi_{\beta_3}\beta_3 &= g_3 V.\tau_3\beta_3.\end{aligned}\right\} \tag{39}$$

The conditions

$$\phi_{\beta_1}\tau_1 \doteq \phi_{\beta_2}\tau_2 = \phi_{\beta_3}\tau_3 = 0 \tag{40}$$

yield three other vector equations.

Two more scalar conditions can be obtained from the fact that we must have

$$\beta_1 = \beta_2 - \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_2}\beta_3}\beta_3$$

and

$$\left.\beta_2 = \beta_3 - \frac{S.\beta_1\phi_{\beta_2}\beta_3}{S.\beta_3\phi_{\beta_1}\beta_1}\beta_1,\right\} \tag{41}$$

whereas equation (34) gives

$$\left.\begin{aligned}\beta_1 &= \beta_3 - \omega\beta_2, \\ \beta_2 &= \frac{1}{\omega}(\beta_3 - \beta_1).\end{aligned}\right\} \tag{42}$$

The expressions for β_1 in (41) and (42) can differ only by a factor, and we conclude that

$$\beta_3 - \omega\beta_2 = x\left(\beta_2 - \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_2}\beta_3}\beta_3\right);$$

whence
$$x = -\omega\,;\quad \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\phi_{\beta_2}\beta_3} = \frac{1}{\omega} = \omega^2. \tag{43}$$

Similarly,
$$\frac{S.\beta_1\phi_{\beta_1}\beta_3}{S.\beta_3\phi_{\beta_1}\beta_1} = 1. \tag{44}$$

We shall now put τ_1, τ_2, τ_3 in the forms

$$\left.\begin{aligned} \tau_1 &= x_1\rho_1 + x_2\rho_2 + x_3\rho_3, \\ \tau_2 &= y_1\rho_1 + y_2\rho_2 + y_3\rho_3, \\ \tau_3 &= z_1\rho_1 + z_2\rho_2 + z_3\rho_3, \end{aligned}\right\} \tag{45}$$

and equations (39) become

$$\left.\begin{aligned} \phi_{\rho_1}\rho_1 + c^2\phi_{\rho_2}\rho_2 - 2c\phi_{\rho_1}\rho_2 &= [-(cx_1 + x_2)V.\rho_1\rho_2 + x_3 V.\rho_3\rho_1 + cx_3 V.\rho_2\rho_3]\,g_1, \\ \phi_{\rho_1}\rho_1 + \omega^2c^2\phi_{\rho_2}\rho_2 - 2\omega c\phi_{\rho_1}\rho_2 &= [-(\omega cy_1 + y_2)V.\rho_1\rho_2 + y_3 V.\rho_3\rho_1 + \omega cy_3 V.\rho_2\rho_3]\,g_2, \\ \phi_{\rho_1}\rho_1 + \omega c^2\phi_{\rho_2}\rho_2 - 2\omega^2c\phi_{\rho_1}\rho_2 &= [-(\omega^2cz_1 + z_2)V.\rho_1\rho_2 + z_3 V.\rho_3\rho_1 + \omega^2cz_3 V.\rho_2\rho_3]\,g_3. \end{aligned}\right\} \tag{46}$$

While from equations (40) we obtain

$$\left.\begin{aligned} x_1\phi_{\rho_1}\rho_1 - cx_2\phi_{\rho_2}\rho_2 + (x_2 - cx_1)\,\phi_{\rho_1}\rho_2 + x_3\phi_{\rho_1}\rho_3 - cx_3\rho_{\rho_2}\rho_3 &= 0, \\ y_1\phi_{\rho_1}\rho_1 - \omega cy_2\phi_{\rho_2}\rho_2 + (y_2 - \omega cy_1)\,\phi_{\rho_1}\rho_2 + y_3\phi_{\rho_1}\rho_3 - \omega cy_3\phi_{\rho_2}\rho_3 &= 0, \\ z_1\phi_{\rho_1}\rho_1 - \omega^2cz_2\phi_{\rho_2}\rho_2 + (z_2 - \omega^2cz_1)\,\phi_{\rho_1}\rho_2 + z_3\phi_{\rho_1}\rho_3 - \omega^2cz_3\phi_{\rho_2}\rho_3 &= 0. \end{aligned}\right\} \tag{47}$$

The fact that $\phi_{\beta_1}\beta_2$, $\phi_{\beta_2}\beta_3$, $\phi_{\beta_3}\beta_1$ are respectively multiples of $V.\tau_1\tau_2$, $V.\tau_2\tau_3$, $V.\tau_3\tau_1$ leads to the three following vector equations, in which h_1, h_2, h_3 are scalars:

$$\left.\begin{aligned} &\phi_{\rho_1}\rho_1 - (c + \omega c)\,\phi_{\rho_1}\rho_2 + \omega c^2\phi_{\rho_2}\rho_2 \\ &\qquad = [(x_1y_2 - x_2y_1)V.\rho_1\rho_2 + (x_2y_3 - x_3y_2)V.\rho_2\rho_3 + (x_3y_1 - x_1y_3)V.\rho_3\rho_1]\,h_1, \\ &\phi_{\rho_1}\rho_1 - (\omega c + \omega^2c)\,\phi_{\rho_1}\rho_2 + c^2\phi_{\rho_2}\rho_2 \\ &\qquad = [(y_1z_2 - y_2z_1)V.\rho_1\rho_2 + (y_2z_3 - y_3z_2)V.\rho_2\rho_3 + (y_3z_1 - y_1z_3)V.\rho_3\rho_1]\,h_2, \\ &\phi_{\rho_1}\rho_1 - (\omega^2c + c)\,\phi_{\rho_1}\rho_2 + \omega^2c^2\phi_{\rho_2}\rho_2 \\ &\qquad = [(z_1x_2 - z_2x_1)V.\rho_1\rho_2 + (z_2x_3 - z_3x_2)V.\rho_2\rho_3 + (z_3x_1 - z_1x_3)V.\rho_3\rho_1]\,h_3. \end{aligned}\right\} \tag{48}$$

There will be no loss of generality by making

$$x_3 = y_3 = z_3 = 1 \tag{49}$$

in these equations. After doing this, multiply the first of equations (47) by 1, the second by ω and the third by ω^2 and add the results. By this process is obtained the vector equation

$$\left.\begin{aligned} &\phi_{\rho_1}\rho_1(x_1 + \omega y_1 + \omega^2z_1) - \phi_{\rho_2}\rho_2(x_2 + \omega^2y_2 + \omega z_2)\,c \\ &\qquad + \phi_{\rho_1}\rho_2[x_2 + \omega y_2 + \omega^2z_2 - c(x_1 + \omega^2y_1 + \omega z_1)] = 0, \end{aligned}\right\} \tag{50}$$

which yields the three scalar equations,

$$\left.\begin{aligned} x_1 + \omega y_1 + \omega^2 z_1 &= 0, \\ x_2 + \omega^2 y_2 + \omega z_2 &= 0, \\ x_2 + \omega y_2 + \omega^2 z_2 - c(x_1 + \omega^2 y_1 + \omega z_1) &= 0, \end{aligned}\right\} \tag{51}$$

unless in the exceptional case when

$$S.\phi_{\rho_1}\rho_1\ \phi_{\rho_2}\rho_2\ \phi_{\rho_1}\rho_2 = 0.$$

If we add the three equations (46), the sum of the left members is $3\phi_{\rho_1}\rho_1$, which is also the sum of the left numbers of (48); we may therefore equate the right members and thus obtain three more scalar equations:

$$\left.\begin{aligned} -g_1(cx_1 + x_2) - g_2(\omega c y_1 + y_2) - g_3(\omega^2 c z_1 + z_2)& \\ = h_1(x_1y_2 - x_2y_1) + h_2(y_1z_2 - y_2z_1) + h_3(z_1x_2 - z_2x_1)&, \\ g_1 + g_2 + g_3 = h_1(x_2 - y_2) + h_2(y_2 - z_2) + h_3(z_2 - x_2)&, \\ cg_1 + \omega c g_2 + \omega^2 c g_3 = h_1(y_1 - x_1)\, h_2(z_1 - y_1) + h_3(x_1 - z_1)&. \end{aligned}\right\} \tag{52}$$

Again in (47) and (48) multiply the first, second and third equations in each group respectively by $1, \omega, \omega^2$; add and compare results. Three other scalar equations are obtained—

$$\left.\begin{aligned} -\omega g_1(cx_1 + x_2) - \omega^2 g_2(\omega c y_1 + y_2) - g_3(\omega^2 c z_1 + z_2)& \\ = h_1(x_1y_2 - x_2y_1) + \omega h_2(y_1z_2 - y_2z_1) + \omega^2 h_3(z_1x_2 - z_2x_1)&, \\ \omega g_1 + \omega^2 g_2 + g_3 = h_1(x_2 - y_2) + \omega h_2(y_2 - z_2) + \omega^2 h_3(z_2 - x_2)&, \\ c\omega g_1 + cg_2 + c\omega^2 g_3 = h_1(y_1 - x_1) + \omega h_2(z_1 - y_1) + \omega^2 h_3(x_1 - z_1)&. \end{aligned}\right\} \tag{53}$$

Finally, using $\omega^2, \omega, 1$ as multipliers, adding and comparing, we derive three more scalar equations:

$$\left.\begin{aligned} \omega^2 g_1(cx_1 + x_2) + \omega g_2(\omega c y_1 + y_2) + g_3(\omega^2 c z_1 + z_2)& \\ = 2h_1(x_1y_2 - x_2y_1) + 2\omega^2 h_2(y_1z_2 - y_2z_1) + 2\omega h_3(z_1x_2 - z_2x_1)&, \\ -\omega^2 g_1 - \omega g_2 - g_3 = 2h_1(x_2 - y_2) + 2\omega^2 h_2(y_2 - z_2) + 2\omega h_3(z_2 - x_2)&, \\ -\omega^2 c g_1 - \omega^2 c g_2 - \omega^2 c g_3 = 2h_1(y_1 - x_1) + 2\omega^2 h_2(z_1 - y_1) + 2\omega h_3(x_1 - z_1)&. \end{aligned}\right\} \tag{54}$$

Equations (51), (52), (53), and (54), twelve in number, enable us to determine eleven of the unknown quantities

$$\begin{matrix} x_1, & y_1, & z_1, \\ x_2, & y_2, & z_2, \\ g_1, & g_2, & g_3, \\ h_1, & h_2, & h_3 \end{matrix}$$

in terms of the twelfth one.

To obtain the complete solution of these equations, which we know to be consistent, is apart from the present purpose. It will suffice to know a single system of values of the unknown quantities, and, taking a hint from equations (26), let us inquire if

$$\left.\begin{aligned} x_1 &= ca_1, \quad y_1 = c\omega a_1, \quad z_1 = c\omega^2 a_1, \\ x_2 &= \frac{a_2}{c}, \quad y_2 = \frac{\omega^2 a_2}{c}, \quad z_2 = \frac{\omega a_2}{c}, \end{aligned}\right\} \tag{55}$$

where a_1 and a_2 are to be determined, are possible forms for these quantities. Substituting them in equations (51), the first two are identically satisfied, while the third becomes

$$\frac{3a_2}{c} - 3c^2 a_1 = 0,$$

or

$$a_2 = c^3 a_1, \tag{56}$$

as we should have expected from equation (25). The last two equations in (52), (53) and (54) become respectively

$$\left.\begin{aligned} g_1 + g_2 + g_3 &= a_1 [h_1(\omega - 1) + h_2(\omega^2 - \omega) + h_3(1 - \omega^2)], \\ g_1 + \omega g_2 + \omega^2 g_3 &= \frac{a_2}{c} [h_1(1 - \omega^2) + h_2(\omega^2 - \omega) + h_3(\omega - 1)], \end{aligned}\right\} \tag{52'}$$

$$\left.\begin{aligned} \omega g_1 + \omega^2 g_2 + g_3 &= a_1 [h_1(\omega - 1) + h_2(1 - \omega^2) + h_3(\omega^2 - \omega)], \\ \omega g_1 + g_2 + \omega^2 g_3 &= \frac{a_3}{c} [h_1(1 - \omega^2) + h_2(1 - \omega^2) + h_3(1 - \omega^2)], \end{aligned}\right\} \tag{53'}$$

$$\left.\begin{aligned} -(\omega^2 g_1 + \omega g_2 + g_3) &= \frac{2a_2}{c} [h_1(1 - \omega^2) + h_2(\omega - 1) + h_3(\omega^2 - \omega)], \\ -\omega^2 (g_1 + g_2 + g_3) &= 2a_1 [h_1(\omega - 1) + h_2(\omega - 1) + h_3(\omega - 1)]. \end{aligned}\right\} \tag{54'}$$

The values of $g_1, \ldots, h_3$ can now be found from equations (33), (43) and (44), together with

$$\left.\begin{aligned} \phi_{\beta_1}\beta_2 &= h_1 V.\tau_1\tau_2 \\ -\omega\phi_{\beta_2}\beta_3 &= h_2 V.\tau_2\tau_3 \\ -\omega\phi_{\beta_3}\beta_1 &= h_3 V.\tau_3\tau_1. \end{aligned}\right\} \tag{48'}$$

From equations (33) and (46) we find

$$-\omega = \frac{g_1(cx_3 - \omega cx_3)}{g_2(\omega cy_3 - cy_3)} = -\frac{g_1}{g_2}. \tag{58}$$

From (43) and (46), together with (35),

$$\omega^2 = \frac{-\frac{1}{\omega}\, g_2(\omega c y_3 - \omega^2 c y_3)}{\frac{1}{\omega^2}\, g_3(\omega^2 z_3 - \omega c z_3)} = +\,\omega\,\frac{g_2}{g_3}. \tag{59}$$

From (44) and (46),

$$1 = \frac{\frac{1}{\omega^2}\, g_3(\omega^2 c z_3 - c z_3)}{-\frac{1}{\omega}\, g_1(c x_3 - \omega^2 c x_3)} = +\,\omega^2\,\frac{g_3}{g_1}. \tag{60}$$

Equations (58), (59), (60) are consistent without regard to c; we have therefore

$$\frac{g_2}{g_1} = \omega^2;\quad \frac{g_3}{g_1} = \omega. \tag{61}$$

From (48′) we find that

$$h_1 = \frac{S.\beta_2\phi_{\beta_1}\beta_1}{S.\beta_1\tau_1\tau_2} = \frac{g_1(c x_3 - \omega c x_3)}{x_2 y_3 - y_2 x_3 + c(x_1 y_3 - x_3 y_1)}$$

$$-\frac{1}{\omega}\,h_2 = \frac{S.\beta_3\phi_{\beta_2}\beta_2}{S.\beta_2\tau_2\tau_3} = \frac{-\frac{1}{\omega}\, g_2(\omega c y_3 - \omega^2 c y_3)}{y_2 z_3 - z_2 y_3 + \omega c(y_1 z_3 - y_3 z_1)}$$

$$-\frac{1}{\omega}\,h_3 = \frac{S.\beta_1\phi_{\beta_3}\beta_3}{S.\beta_3\tau_3\tau_1} = \frac{\frac{1}{\omega^2}\, g_3(\omega^2 c z_3 - c z_3)}{-\frac{1}{\omega}\,[z_2 x_3 - x_2 z_3 + \omega^2 c(z_1 x_3 - z_3 x_1)},$$

or more simply,

$$h_1 = \frac{g_1 c(1 - \omega)}{x_2 - y_2 + c(x_1 - y_1)},$$

$$h_2 = \frac{-g_2 c\omega(\omega - 1)}{y_2 - z_2 + \omega c(y_1 - z_1)},$$

$$h_3 = \frac{g_3 c(\omega + 1)(\omega - 1)}{z_2 - x_2 + \omega^2 c(z_1 - x_1)}.$$

By substituting here the values (55) assumed for $x_1, \ldots, z_2$, we have

$$\left.\begin{aligned} h_1 &= \frac{g_1 c^2(1 - \omega)}{3a_2}, \\ h_2 &= \frac{-g_2 c^2(\omega - 1)\,\omega^2}{3a_2}, \\ h_3 &= \frac{-g_3 c^2(\omega - 1)\,\omega}{3a_2}. \end{aligned}\right\} \tag{62}$$

We conclude that

$$\left.\begin{aligned} \frac{a_2}{c}\,\frac{h_1}{g_1} &= \frac{c}{3}\,(1-\omega), \\ \frac{a_2}{c}\,\frac{h_2}{g_1} &= -\frac{c}{3}(\omega-1)\,\omega = \frac{c}{3}\,\omega(1-\omega), \\ \frac{a_2}{c}\,\frac{h_3}{g_1} &= -\frac{c}{3}\,\omega^2(\omega-1) = \frac{c}{3}\,\omega^2(1-\omega). \end{aligned}\right\} \tag{63}$$

With these values, equations (52′), (53′), (54′) are all verified, provided that we impose upon c the condition

$$c = 1. \tag{64}$$

This leads to the conclusion that, equation (56),

$$a_1 = a_2, \tag{65}$$

while g_1 is undetermined. We have thus

$$\left.\begin{aligned} \frac{h_1}{g_1} &= \frac{1}{3a_2}(1-\omega), \\ \frac{h_2}{g_1} &= \frac{1}{3a_2}\,\omega(1-\omega), \\ \frac{h_3}{g_1} &= \frac{1}{3a_2}\,\omega^2(1-\omega). \end{aligned}\right\} \tag{66}$$

By these values of $g_2, \ldots, h_3, c$, the first equations of (52), (53) and (54) are reduced to identities without regard to the values of g_1 and a_2.

Let us now return to equations (46) and add them together, keeping in mind the values found for the scalar coefficients. We thus obtain

$$\phi_{\rho_1}\rho_1 = g_1 V.\rho_2\rho_3. \tag{67}$$

Again add them, after multiplying them in order by $1, \omega, \omega^2$, and we have

$$\phi_{\rho_2}\rho_2 = g_1 V.\rho_3\rho_1. \tag{68}$$

Finally, multiply by $\omega^2, \omega, 1$ and add, the result is

$$\phi_{\rho_1}\rho_2 = a_2 g_1 V.\rho_1\rho_2. \tag{69}$$

Adding equations (47) and using (68), we find

$$\phi_{\rho_1}\rho_3 = a_2\phi_{\rho_2}\rho_2 = a_2 g_1 V.\rho_3\rho_1. \tag{70}$$

Multiply equations (47) in order by $1, \omega^2, \omega$ and add. This gives

$$\phi_{\rho_2}\rho_3 = a_2\phi_{\rho_1}\rho_1 = a_2 g_1 V.\rho_2\rho_3. \tag{71}$$

From (70) and (71) we learn that

$$S.\rho_1\phi_{\rho_2}\rho_3 = S.\rho_2\phi_{\rho_2}\rho_3 = 0,$$

and that consequently

$$\phi_{\rho_2}\rho_3 = x V.\rho_1\rho_2. \tag{72}$$

Since $V.\rho_1\rho_2$, $V.\rho_2\rho_3$, $V.\rho_3\rho_1$ are respectively proportional to ρ_3, ρ_1, ρ_2, the existence of three numbers having the property defined by equations (18) is demonstrated. The points ρ_1, ρ_2, ρ_3 form a triangle upon each side of which lie three points of inflection. Their affixes are given by the following scheme:

$$\left.\begin{aligned} \beta_1 &= \rho_1 - \rho_2, \\ \beta_2 &= \rho_1 - \omega\rho_2, \\ \beta_3 &= -\frac{1}{\omega}(\rho_1 - \omega^2\rho_2), \end{aligned}\right\}$$

on the side joining ρ_1 and ρ_2.

$$\left.\begin{aligned} \gamma^1 &= \rho_2 - \rho_3, \\ \gamma^2 &= \rho_2 - \omega\rho_3, \\ \gamma_3 &= -\frac{1}{\omega}(\rho_2 - \omega^2\rho_3), \end{aligned}\right\}$$

on the side joining ρ_2 and ρ_3.

$$\left.\begin{aligned} \delta_1 &= \rho_3 - \rho_1, \\ \delta_2 &= \rho_3 - \omega\rho_1, \\ \delta_3 &= -\frac{1}{\omega}(\rho_3 - \omega^2\rho_1), \end{aligned}\right\}$$

on the side joining ρ_3 and ρ_1.

We find by solving these equations that

$$\left.\begin{aligned} \rho_1 &= \tfrac{1}{3}(\beta_1 + \beta_2 - \omega\beta_3), \\ \rho_2 &= \tfrac{1}{3}(\gamma_1 + \gamma_2 - \omega\gamma_3), \\ \rho_3 &= \tfrac{1}{3}(\delta_1 + \delta_2 - \omega\delta_3). \end{aligned}\right\} \tag{73}$$

It is to be concluded that when the arrangement of the points of inflection on the sides of a triangle is known, the affixes of the vertices will be given by equations similar to (73), and those affixes will have the properties given in equations (18). It follows that the equation

$$\phi_\rho\rho = x\rho \tag{74}$$

has at least twelve solutions which fall into four sets with three members in a set.

In equation (72) the scalar x is undetermined, but its value is easily found. There is a number $\sigma_1 = l\rho_1 + m\rho_2 + n\rho_3$ which satisfies the equation

$$\phi_{\gamma_1}\sigma_1 = 0. \tag{75}$$

This gives us without difficulty, since $\gamma_1 = \rho_2 - \rho_3$,

$$\left.\begin{aligned} la_2g_1 - nx &= 0, \\ m - la_2 &= 0, \\ n - m &= 0, \end{aligned}\right\} \tag{76}$$

whence

$$x = g_1. \tag{77}$$

The equation of the cubic referred to the triangle ρ_1, ρ_2, ρ_3 takes the form

$$x^3 + y^3 + z^3 + 6a_2xyz = 0, \tag{78}$$

and we conclude that a_2 is the absolute invariant of the ternary cubic.*

From equations (67), (68), (72) we have

$$S.\rho_1\phi_{\rho_1}\rho_1 = S.\rho_2\phi_{\rho_2}\rho_2 = S.\rho_3\phi_{\rho_3}\rho_3 = g_1S.\rho_1\rho_2\rho_3. \tag{79}$$

Taking $\alpha = x\rho_1$, $\beta = y\rho_2$, $\gamma = z\rho_3$, we shall have

$$\phi_{\rho_1}\rho_1 = \frac{1}{x^2}\,\phi_\alpha\alpha = g_1V.\rho_2\rho_3 = \frac{g_1}{yz}\,V.\beta\gamma\,;$$

whence

$$\left.\begin{aligned} \phi_\alpha\alpha &= \frac{g_1x^2}{yz}\,V.\beta\gamma, \\ \phi_\beta\beta &= \frac{g_1y^2}{zx}\,V.\gamma\alpha, \\ \phi_\gamma\gamma &= \frac{g_1z^2}{xy}\,V.\alpha\beta. \end{aligned}\right\} \tag{80}$$

and similarly,

It will be seen that any one of the choices

$$\left.\begin{aligned} &x,\ y = \omega x\ ,\ z = \omega^2x, \\ &x,\ y = \omega^2x,\ z = \omega x\ , \\ &x,\ y = x\ \ ,\ z = x \end{aligned}\right\} \tag{81}$$

preserves the property of equations (79),

$$S.\alpha\phi_\alpha\alpha = S.\beta\phi_\beta\beta = S.\gamma\phi_\gamma\gamma. \tag{82}$$

JOHNS HOPKINS UNIVERSITY, *December* 30, 1891.

---

* Cf. Clebsch, Vorlesungen, p. 510.

On the Part of the Parallactic Inequalities in the Moon's Motion which is a Function of the Mean Motions of the Sun and Moon.

By Ernest W. Brown, M. A., *Haverford College, Pa.*

In Vol. I of the *American Journal of Mathematics*,[*] Mr. G. W. Hill, taking into account only inequalities depending on the mean motions of the sun and moon, has shown that these inequalities are able to be determined to a high degree of accuracy by using moving rectangular axes, one of which passes through the mean place of the sun. This paper is an adaptation of his method so as to include that class of inequalities which depends also on the ratio of the solar and lunar distances, and in particular, the principal part of the Parallactic Inequality. Owing to the use which has been made of this latter in obtaining the parallax of the sun, it becomes of importance that its coefficient should be accurately known. Delaunay's expressions[†] are deficient in this respect, and we have no means of knowing how near Hansen's numerical value is to the truth. The inequalities are obtained below in an algebraical and numerical form, the latter giving their coefficients in longitude and parallax true to about one thousandth of a second of arc.

I.

Transformation of the Equations of Motion.

The inequalities from purely circular motion which depend only on the lunar eccentricity and mean motions of the sun and moon only, are given by the equations

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x &= 0, \\ \frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}\,y &= 0, \end{aligned}\right\} \qquad (1)$$

---

[*] Researches in the Lunar Theory, pp. 5, 129, 245.

[†] Mém. Fr. Acad. Sc., Vol. XXIX, p. 847.

where n', μ have their usual significations, the axes being rectangular and revolving with uniform angular velocity n', and that of x passing through the mean sun. The earth is supposed to be moving in a circle round the sun with uniform angular velocity n'. The above equation is that which would be obtained from the consideration of a disturbing body of infinitely great mass, and at an infinite distance moving round the earth in a circle with finite uniform angular velocity n', where $n'^2 = \text{mass} \div (\text{dist.})^3$. In order to get the inequalities free from the lunar eccentricity, Mr. Hill gets a particular solution of the above equations. When now we wish to include inequalities dependent on the distance of the sun, we can no longer suppose it at an infinite distance and of infinite mass. We must include the part of the disturbing function due to this cause which has been omitted. It will then be seen that we can obtain all the inequalities which a disturbing body moving round the earth with constant angular velocity in the moon's orbit produces, when the moon's undisturbed orbit is taken to be circular.

Hence we have, instead of zero on the right-hand sides of equations (1), to put $\frac{d\Omega_1}{dx}$ and $\frac{d\Omega_1}{dy}$ respectively, where

$$\Omega_1 = \frac{n'^2}{a'}\left[x^3 - \frac{3}{2}xy^2\right] + \frac{n'^2}{a'^2}\left[x^4 - 3x^2y^2 + \frac{3}{8}y^4\right] \\ + \frac{n'^2}{a'^3}\left[x^5 - 5x^3y^2 + \frac{15}{8}xy^4\right] + \ldots .$$

Multiplying the equations thus formed by $\frac{dx}{dt}$, $\frac{dy}{dt}$, adding and integrating the result, we get the Jacobian equation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - \frac{2\mu}{r} - 3n'^2x^2 = 2\Omega_1 - 2C.$$

Transforming to new coordinates u, s, where

$$u = x + y\sqrt{-1},$$
$$s = x - y\sqrt{-1},$$

our equations of motion and the Jacobian integral become respectively

$$\left.\begin{aligned} &\frac{d^2u}{dt^2} + 2n'\sqrt{-1}\,.\,\frac{du}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}u = \frac{3}{2}n'^2(u+s) + 2\frac{d\Omega_1}{ds}, \\ &\frac{d^2s}{dt^2} - 2n'\sqrt{-1}\,.\,\frac{ds}{dt} + \frac{\mu}{(us)^{\frac{3}{2}}}\,.\,s = \frac{3}{2}n'^2(u+s) + 2\frac{d\Omega_1}{du}, \\ &\frac{du}{dt}\cdot\frac{ds}{dt} = \frac{2\mu}{(us)^{\frac{1}{2}}} + \frac{3}{4}n'^2(u+s)^2 + 2\Omega_1 - 2C, \end{aligned}\right\} \qquad (2)$$

and Ω_1 takes the form

$$
\begin{aligned}
\Omega_1 = \frac{n'^2}{a'}\Big[\frac{5}{16}(u^3+s^3)+\frac{3}{16}us(u+s)\Big] \\
+\frac{n'^2}{a'^2}\Big[\frac{35}{128}(u^4+s^4)+\frac{5}{32}us(u^2+s^2)+\frac{9}{64}u^2s^2\Big] \\
+\frac{n'^2}{a'^3}\Big[\frac{63}{256}(u^5+s^5)+\frac{35}{256}us(u^3+s^3)+\frac{15}{128}u^2s^2(u+s)\Big] \\
+\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;.\qquad
\end{aligned}
$$

Ω_1, expressed in terms of u, s, is obtained directly by expanding $\dfrac{n'^2a'^3}{\rho}$, where ρ is the distance between the sun and moon and is equal in this case to

$$\sqrt{(a'^2 - 2a'r\cos\theta + r^2)}.$$

For since $r^2 = us$ and $2r\cos\theta = 2x = u+s$, we have $2r^n\cos n\theta = u^n + s^n$, and therefore

$$\frac{n'^2a'^3}{\rho} = n'^2a'^2\Big(1+\frac{r}{a'}P_1+\frac{r^2}{a'^2}P_2+\ldots.\Big).$$

P_n is the zonal harmonic of degree n, and is expressible in terms of

$$\cos n\theta,\ \cos(n-2)\,\theta\ldots.$$

Hence we have

$$\Omega_1 = r^3\frac{n'^2}{a'}P_3 + r^4\frac{n'^2}{a'^2}P_4+\ldots.,$$

where

$$r^nP_n = \frac{1.3\ldots.(2n-1)}{2.4\ldots.2n}(u^n+s^n)+\frac{1.3\ldots.(2n-3)}{2.4\ldots.(2n-2)}\cdot\frac{1}{2}us(u^{n-2}+s^{n-2})+\ldots.$$

II.

Solution of the Equations.

In order to solve our equations for the particular class of inequalities which we wish to get, take the particular integrals

$$
\begin{aligned}
x &= \Sigma A_i\cos i\nu(t-t_0),\\
y &= \Sigma A_i\sin i\nu(t-t_0),
\end{aligned}
$$

i having positive integral values from zero to infinity, and $\nu = n - n'$. Put

$$A_i = a_{i-1}+a_{-i-1},\quad B_i = a_{i-1}-a_{-i-1},$$

we obtain

$$
\begin{aligned}
x &= \Sigma a_{i-1}\cos i\nu(t-t_0),\\
y &= \Sigma a_{i-1}\sin i\nu(t-t_0),
\end{aligned}
$$

where the summation is now extended to negative values of i. Transforming to the complex variables u, s, and putting $e^{\nu(t-t_0)\sqrt{-1}}=\zeta$, we have

$$u=\Sigma a_{i-1}\zeta^i,$$
$$s=\Sigma a_{-i-1}\zeta^i.$$

Also putting

$$m=\frac{n'}{\nu}\,,\quad \varkappa=\frac{\mu}{\nu^2}\,,\quad \zeta\frac{d}{d\zeta}=-\frac{\sqrt{-1}}{\nu}\cdot\frac{d}{dt}=D,$$

the equations (2) may be written

$$\left[D^2+2mD-\frac{\varkappa}{(us)^{\frac{3}{2}}}\right]u=-\frac{3}{2}m^2(u+s)-m^2.\frac{2}{n'^2}\cdot\frac{d\Omega_1}{ds}\,,$$
$$\left[D^2-2mD-\frac{\varkappa}{(us)^{\frac{3}{2}}}\right]s=-\frac{3}{2}m^2(u+s)-m^2.\frac{2}{n'^2}\cdot\frac{d\Omega_1}{du}\,,$$
$$Du.Ds+\frac{2\varkappa}{(us)^{\frac{1}{2}}}=-\frac{3}{4}m^2(u+s)^2-m^2.\frac{2}{n'^2}\,.\,\Omega_1+C'.$$

Multiply the first of these equations by s, the second by u, and add to the last; also with the same multipliers subtract the first from the second, the resulting equations will be

$$\left.\begin{aligned}D^2(us)-DuDs-2m\,(uDs-sDu)+\frac{9}{4}m^2(u+s)^2&\\ =-m^2.\frac{2}{n'^2}\left(u\frac{d\Omega_1}{du}+s\frac{d\Omega_1}{ds}+\Omega_1\right)+C&,\\ D\,(uDs-sDu-2mus)+\frac{3}{2}m^2(u^2-s^2)=-m^2.\frac{2}{n'^2}\left(u\frac{d\Omega_1}{du}-s\frac{d\Omega_1}{ds}\right)&.\end{aligned}\right\}(3)$$

The constant $\varkappa$ has vanished and will have to be determined from our previous equations in terms of the new constants introduced in the Particular Solution (Hill, p. 132).

Substituting in (3) the values of u, s, which are

$$u=\Sigma_i a_{i-1}\zeta^i,\qquad s=\Sigma_i a_{-i-1}\zeta^i,$$

or, what is the same thing,

$$u=\Sigma_i a_i\zeta^{i+1},\qquad s=\Sigma_i a_{-i}\zeta^{i-1},$$

so that

$$us=\Sigma_i\Sigma_j a_j a_{j-i}\zeta^i,\qquad uDs-sDu=\Sigma_i\Sigma_j\,(i-2j-2)\,a_j a_{j-i}\zeta^i,\ \text{etc.,}$$

and equating the coefficient of ζ^i to zero (except when $i=0$), we obtain

$$\left.\begin{aligned}
&\Sigma_j \left[i^2-(j+1)(i-j-1)-2m(i-2j-2)+\frac{9}{2}m^2\right] a_j a_{j-i}\\
&\qquad\qquad +\frac{9}{4}m^2\Sigma_j\left[a_{j-1}a_{i-j-1}+a_{-j-1}a_{-i+j-1}\right]=-m^2L_i,\\
&\Sigma_j\left[i(i-2j-2)-2mi\right]a_j a_{j-i}\\
&\qquad\qquad +\frac{3}{2}m^2\Sigma_j\left[a_{j-1}a_{i-j-1}-a_{-j-1}a_{-i+j-1}\right]=-m^2M_i,
\end{aligned}\right\}\quad(4)$$

where $-m^2L_i$, $-m^2M_i$ are the coefficients of ζ^i on the right-hand sides of equations (3). Multiply these equations by 2 and 3 respectively, and take the sum and difference, we get

$$\begin{aligned}
&\Sigma_j\left[5i^2-8(j+1)i+2(j+1)^2-2m(5i-4j-4)+9m^2\right]a_j a_{-i+j}\\
&\qquad\qquad +9m^2\Sigma_j a_{j-1}a_{i-j-1} \quad =-m^2(2L_i+3M_i),\\
&\Sigma_j\left[-i^2+4(j+1)i+2(j+1)^2+2m(i+4j+4)+9m^2\right]a_j a_{-i+j}\\
&\qquad\qquad +9m^2\Sigma_j a_{-j-1}a_{-i+j-1}=-m^2(2L_i-3M_i).
\end{aligned}$$

The terms of lowest order in these equations are a_0a_i and a_0a_{-i}. In order to separate out these, multiply the last pair of equations by

$$\begin{aligned}
&-i^2+4i+2+2m(i+4)+9m^2,\\
&-5i^2+8i-2+2m(5i-4)-9m^2,
\end{aligned}$$

respectively, add the products and divide the whole by

$$12i^2\left[2(i^2-1)-4m+m^2\right].$$

In the resulting equation the term a_0a_{-i} will have vanished and the coefficient of a_0a_i will be -1. The equation is

$$\left.\begin{aligned}
&\Sigma_j\left\{[i,j]\,a_j a_{-i+j}+[i]\,a_{j-1}a_{i-j-1}+(i)\,a_{-j-1}a_{-i+j-1}\right\}\\
&\qquad\qquad =-\frac{1}{9}\left[2L_i\{[i]+(i)\}+3M_i\{[i]-(i)\}\right],
\end{aligned}\right\}\ (5)$$

where

$$\begin{aligned}
[i,j]&=-\frac{j}{i}\cdot\frac{(i-2)j+i^2+2i-2-2(j-i+2)m+m^2}{2(i^2-1)-4m+m^2},\\
[i]&=-\frac{3m^2}{4i^2}\,\frac{i^2-4i-2-2(i+4)m-9m^2}{2(i^2-1)-4m+m^2},\\
(i)&=-\frac{3m^2}{4i^2}\,\frac{5i^2-8i+2-2(5i-4)m+9m^2}{2(i^2-1)-4m+m^2},
\end{aligned}$$

and therefore, as they should be,

$$[i, i] = -1 \qquad [i, 0] = 0.$$

The equation (5) corresponds to that obtained by Mr. Hill on p. 135 of his memoir referred to. It may be shown that if we put zero instead of the function on the right-hand side of equation (5), every a with an odd suffix will vanish, and with one or two changes in notation it becomes the same as his.

The expressions L_i, M_i must now be obtained. Since Ω_1 is formed of homogeneous functions of u, s of the 3d, 4th, ... degrees, we have

$$\Omega_1 + u\frac{d\Omega_1}{du} + s\frac{d\Omega_1}{ds} = 4\,.\,\frac{n'^2}{a'}\left[\frac{5}{16}(u^3+s^3) + \frac{3}{16}us(u+s)\right]$$
$$+ 5\,.\,\frac{n'^2}{a'^2}\left[\frac{35}{128}(u^4+s^4) + \frac{5}{32}us(u^2+s^2) + \frac{9}{64}u^2s^2\right]$$
$$+ 6\,.\,\frac{n'^2}{a'^3}\left[\frac{63}{256}(u^5+s^5) + \frac{35}{256}us(u^3+s^3) + \frac{15}{128}u^2s^2(u+s)\right] + \ldots\ldots,$$
$$u\frac{d\Omega_1}{du} - s\frac{d\Omega_1}{ds} = \frac{n'^2}{a'}\left[\frac{15}{16}(u^3-s^3) + \frac{3}{16}us(u-s)\right]$$
$$+ \frac{n'^2}{a'^2}\left[\frac{35}{32}(u^4-s^4) + \frac{5}{16}us(u^2-s^2)\right]$$
$$+ \frac{n'^2}{a'^3}\left[\frac{315}{256}(u^5-s^5) + \frac{105}{256}us(u^3-s^3) + \frac{15}{128}u^2s^2(u-s)\right] + \ldots.,$$

and $\frac{n'^2}{2}L_i$, $\frac{n'^2}{2}M_i$ being the coefficients in these when for u, s are substituted their values. Hence

$$-\frac{1}{9}\left[2L_i\{[i]+(i)\} + 3M_i\{[i]-(i)\}\right]$$
$$= -\frac{1}{a'}\left[A_i(u^3)_i + A'_i(s^3)_i + B_i(u^2s)_i + B'_i(us^2)_i\right]$$
$$-\frac{1}{a'^2}\left[C_i(u^4)_i + C'_i(s^4)_i + D_i(u^3s)_i + D'_i(us^3)_i + E_i(u^2s^2)_i\right]$$
$$-\frac{1}{a'^3}\left[F_i(u^5)_i + F'_i(s^5)_i + G_i(u^4s)_i + G'_i(us^4)_i + H_i(u^3s^2)_i + H'_i(u^2s^3)_i\right],$$
$$-\ \ldots\ldots\ldots\ldots$$

where $(u^3)_i$, $(s^3)_i$, etc., denote the coefficients of ζ^i in u^3, s^3, etc. A_i, A'_i, etc. are definite functions of m of the order m^2 at least. Their values are

$$A_i = \frac{5}{72}\{17[i] - (i)\}, \qquad E_i = \frac{5}{16}\{[i] + (i)\},$$

$$B_i = \frac{1}{24}\{11[i] + 5(i)\}, \qquad F_i = \frac{21}{128}\{9[i] - (i)\},$$

$$C_i = \frac{35}{288}\{11[i] - (i)\}, \qquad G_i = \frac{35}{384}\{7[i] + (i)\},$$

$$D_i = \frac{5}{36}\{4[i] + (i)\}, \qquad H_i = \frac{5}{64}\{5[i] + 3(i)\}.$$

A'_i, B'_i, C'_i, D'_i, F'_i, G'_i, H'_i are got by merely interchanging $[i]$ and (i) in the expressions for the corresponding undashed letters. Hence

$$A_i = -\frac{5m^2}{16i^2}\cdot\frac{2i^2 - 10i - 6 - 4(i+6)m - 27m^2}{2(i^2-1) - 4m + m^2},$$

$$A'_i = -\frac{5m^2}{16i^2}\cdot\frac{14i^2 - 22i + 6 - 4(7i-6)m + 27m^2}{2(i^2-1) - 4m + m^2},$$

$$B_i = -\frac{3m^2}{16i^2}\cdot\frac{6i^2 - 14i - 2 - 4(3i+2)m - 9m^2}{2(i^2-1) - 4m + m^2}.$$

$$B'_i = -\frac{3m^2}{16i^2}\cdot\frac{10i^2 - 18i + 2 - 4(5i-2) + 9m^2}{2(i^2-1) - 4m + m^2},$$

etc.

Also we have

$$L_1 = \frac{2}{a'}\left[\frac{5}{4}\{(u^3)_1 + (s^3)_1\} + \frac{3}{4}\{(u^2s)_1 + (s^2u)_1\}\right] + \ldots\ldots,$$

$$M_1 = \frac{2}{a'}\left[\frac{15}{16}\{(u^3)_1 - (s^3)_1\} + \frac{3}{16}\{(u^2s)_1 - (s^2u)_1\}\right] + \ldots\ldots$$

The coefficients of ζ^i in u^3, s^3, etc., are now to be obtained. From the forms of u, s we have, if the coefficient of ζ^i in u^ps^q be denoted by $(u^ps^q)_i$ as before,

$$(u^ps^q)_i = \Sigma\Sigma\ldots\ldots a_ja_ka_l\ldots\ldots(p \text{ factors}) \times a_sa_ta_w\ldots\ldots(q \text{ factors}),$$

for all integral values, positive and negative, of j, k, $l\ldots\ldots s$, t, $w\ldots\ldots$ consistent with the condition

$$(p + j + k + l\ldots\ldots) - (q + s + t + w\ldots\ldots) = i.$$

From this equation we see that

$$(u^p s^q)_i = (s^p u^q)_{-i}.$$

The coefficient of ζ^i is more easily obtained by taking the coefficient of ζ^h, where

$$h = i - p + q,$$

and therefore

$$(j + k + l + \ldots .) - (s + t + w + \ldots .) = h.$$

III.

The Coefficients a_1 and a_{-1}.

We have now obtained all the expressions necessary for the determination of the coefficients a_i. But before developing them, some remarks must be made on the equations for a_1 and a_{-1}.

It will be noticed that every term in equation (5) except the principal one is divided by the expression $2(i^2 - 1) - 4m + m^2$. When $i = \pm 1$, this reduces to $-4m + m^2$, and the order of a_1 and a_{-1} is thus lowered by one power of m. The consequence of this is a great increase in the difficulty of obtaining their expressions. We have to carry them one order higher to get the same degree of approximation, and when we obtain them by the method of successive approximation, each process, instead of carrying our expressions *two* orders higher, takes them only one; the number of such processes is consequently doubled, and the numerical multipliers of the various powers of m are very much increased in complexity.

A second disadvantage arises from another cause. If we form the expansions of a_1 and a_{-1} in ascending powers of m from equation (5), we find large and continually increasing multipliers of the various powers of m which tend to lessen the value of the successive approximations when expressed numerically, to such an extent that the series, instead of proceeding in a progression whose ratio is roughly m or $1/12$, proceeds in a progression with a ratio of about $5m$ or $2/5$. As this ratio seems somewhat regular between the successive powers, I have used the following method to discover and counteract the effect of the slow convergence. In the following, squares and higher powers of $1/a'$ are neglected.

Going back to the equations (4) obtained directly from the equations of motion, put $i = 1$ in them, and write down the coefficients of a_1, a_{-1}, neglecting terms of the seventh and higher orders. These principal terms then become

$$a_0 a_1 \left[3 + 6m + \frac{9}{2} m^2 + \left(7 + 10m + \frac{9}{2} m^2\right) \frac{a_2}{a_0} + \frac{9}{2} m^2 . \frac{a_{-2}}{a_0}\right]$$
$$+ a_0 a_{-1} \left[1 + 2m + 9m^2 + (1 - 2m + 9m^2) \frac{a_{-2}}{a_0}\right],$$
$$a_0 a_1 \left[- 3 - 2m - (5 + 2m) \frac{a_2}{a_0} + 3m^2 \frac{a_{-2}}{a_0}\right]$$
$$+ a_0 a_{-1} \left[- 1 - 2m + 3m^2 + (1 - 2m + 3m^2) \frac{a_{-2}}{a_0}\right].$$

Now a_2/a_0 and a_{-2}/a_0 involve m only, and their expressions have been found by Mr. Hill. He gives

$$\frac{a_2}{a_0} = \frac{3}{16} m^2 + \frac{1}{2} m^3 + \frac{7}{12} m^4 + \ldots.,$$
$$\frac{a_{-2}}{a_0} = - \frac{19}{16} m^2 - \frac{5}{3} m^3 - \frac{43}{36} m^4 + \ldots.$$

Substituting these values and expanding, we have for the principal terms,

$$a_0 a_1 \left[\ 3 + 6m + \frac{93}{16} m^2 + \ldots.\right] + a_0 a_{-1} \left[\ 1 + 2m + \frac{125}{16} m^2 + \ldots.\right],$$
$$a_0 a_1 \left[- 3 - 2m - \frac{15}{16} m^2 + \ldots.\right] + a_0 a_{-1} \left[- 1 - 2m + \frac{29}{16} m^2 + \ldots.\right].$$

Denote these expressions by $a_0 a_1 P + a_0 a_{-1} P'$ and $a_0 a_1 Q + a_0 a_{-1} Q'$ respectively.

When we wish to separate out a_1 and a_{-1} by the ordinary process of solution, i. e. multiplying the equations by Q', P' and subtracting, also by Q, P and subtracting; the principal terms become

$$a_0 a_1 (PQ' - P'Q),$$
$$a_0 a_{-1} (QP' - Q'P).$$

The resulting expressions for a_1 and a_{-1} thus found would therefore be both divided by the factor $PQ' - Q'P$. Expanding it in powers of m, we find these expressions will be multiplied by

$$- \left(1 - 4m - \frac{37}{8} m^2 - \ldots.\right)^{-1} . \frac{1}{4m},$$

or by

$$- \frac{1}{4m} \left(1 + 4m + \frac{165}{8} m^2 + \ldots.\right).$$

We thus have an explanation of the slow convergence of the series, and by retaining this divisor, we shall be able to obtain series which will converge much more quickly. By proceeding in the following way, the values of a_1 and a_{-1} can be formed to the seventh order in one approximation; and it will also be shown that to this order, our values will be sufficient to obtain the coefficient of this part of the Parallactic Inequality from the expansions, with an error less than two hundredths of a second of arc.

In equations (4) put $i = 1$, we have

$$\begin{aligned}\Sigma_j a_j a_{j-1}\left[j^2 + j + 1 + 2m(2j+1) + \frac{9}{2}m^2\right]& \\ + \Sigma_j \frac{9}{4} m^2 \left[a_{j-1}a_{-j} + a_{-j-1}a_{j-2}\right] &= -m^2 L_1, \\ \Sigma_j a_j a_{j-1}\left[-2j - 1 - 2m\right] \qquad + \Sigma_j \frac{3}{2} m^2 \left[a_{j-1}a_{-j} - a_{-j-1}a_{j-2}\right] &= -m^2 M_1.\end{aligned}$$

Multiply the second equation by $2m$, add to the first and write out the resulting equation and the second equation, for errors of the sixth order at most in m,

$$\left.\begin{aligned}&a_0a_1\left[3 + \frac{m^2}{2} + \frac{a_2}{a_0}\left(7 + \frac{m^2}{2}\right) + \frac{a_{-2}}{a_0}\left(\frac{9}{2}m^2 + 6m^3\right)\right.\\ &\qquad\left. + \frac{a_{-4}}{a_0}\left(\frac{9}{2}m^2 - 6m^3\right)\right] + a_0a_3\left[\frac{a_2}{a_0}\left(13 + \frac{m^2}{2}\right) + \frac{a_4}{a_0}\left(21 + \frac{m^2}{2}\right)\right]\\ &+ a_0a_{-1}\left[1 + 5m^2 + 6m^3 + \frac{a_{-2}}{a_0}(1 + 5m^2 - 6m^3)\right]\\ &+ a_0a_{-3}\left[\frac{9}{2}m^2 - 6m^3 + \frac{a_2}{a_0}\left(\frac{9}{2}m^2 + 6m^3\right) + \frac{a_{-2}}{a_0}\left(3 + \frac{m^2}{2}\right)\right.\\ &\qquad\qquad\left. + \frac{a_{-4}}{a_0}\left(7 + \frac{m^2}{2}\right)\right] = -m^2(L_1 + 2mM_1)\\ &a_0a_1\left[3 + 2m + \frac{a_2}{a_0}(5 + 2m) - 3m^2\frac{a_{-2}}{a_0} + 3m^2\frac{a_{-4}}{a_0}\right]\\ &+ a_0a_3\left[\frac{a_2}{a_0}(7 + 2m) + \frac{a_4}{a_0}(9 + 2m)\right]\\ &+ a_0a_{-1}\left[1 + 2m - 3m^2 + \frac{a_{-2}}{a_0}\left(-1 + 2m + 3m^2\right)\right]\\ &+ a_0a_{-3}\left[3m^2 - 3m^2\frac{a_2}{a_0} + (-3 + 2m)\frac{a_{-2}}{a_0} + (-5 + 2m)\frac{a_{-4}}{a_0}\right] = m^2M_1.\end{aligned}\right\} \quad (6)$$

The equations (6) will be used for a_1 and a_{-1} instead of those obtained from the general form (5).

IV.

The Determination of the Parts of a_i depending on the First Power of the Ratio of the Mean Distances.

The coefficients a_i will now be obtained to the order $m^5.a_0/a'$. To obtain them to this order we have

$$
\begin{aligned}
(u^3)_1 &= 3a_0^2 a_{-2}, & (u^2 s)_1 &= a_0^3 + 2a_0(a_2^2 + a_{-2}^2 + a_2 a_{-2}),\\
(s^3)_1 &= 3a_0^2 a_{-4} + 3a_{-2}^2 a_0, & (us^2)_1 &= a_0^2(2a_{-2} + a_2),\\
(u^3)_3 &= (s^3)_{-3} = a_0^3, & (u^2 s)_3 &= (us^2)_{-3} = a_0^2(2a_2 + a_{-2}),\\
(u^3)_5 &= (s^3)_{-5} = 3a_0^2 a_2, & &
\end{aligned}
$$

the rest of the coefficients of ζ in u^3, s^3, $u^2 s$, $s^2 u$ being zero to the order taken. Whence we have

$$L_1 = \frac{a_0}{a'}\left[\frac{15}{2}(a_0 a_{-2} + a_0 a_{-4} + a_{-2}^2) + \frac{3}{2}(a_0^2 + 2a_2^2 + 2a_{-2}^2 + 2a_2 a_{-2} + 2a_0 a_{-2} + a_0 a_2)\right],$$

$$M_1 = \frac{a_0}{a'}\left[\frac{45}{8}(a_0 a_{-2} - a_0 a_{-4} - a_{-2}^2) + \frac{3}{8}(a_0^2 + 2a_2^2 + 2a_{-2}^2 + 2a_2 a_{-2} - 2a_0 a_{-2} - a_0 a_2)\right]$$

for substitution in the equations (6). Also the right-hand side of equation (5) becomes when

$$
\begin{aligned}
i &= 3, & &\frac{a_0}{a'}[A_3 a_0^2 + B_3 a_0(2a_2 + a_{-2})],\\
i &= -3, & &\frac{a_0}{a'}[A'_{-3} a_0^2 + B'_{-3} a_0(2a_2 + a_{-2})],\\
i &= 5, & &\frac{a_0}{a'} A_5 . 3a_0 a_2,\\
i &= -5, & &\frac{a_0}{a'} A'_{-5} . 3a_0 a_2,
\end{aligned}
$$

whence, writing out equations (5) for a_3, a_{-3}, a_5, a_{-5} with a sufficient number of terms to obtain these quantities as far as the order $m^5.a_0/a'$, we have

$$
\left.
\begin{aligned}
a_0 a_3 = [3,\ 1]\, a_1 a_{-2} + [3,\ 2]\, a_{-1} a_2 + [3]\, 2a_1 a_0 + \frac{a_0}{a'}[A_3 a_0^2 + B_3 a_0(2a_2 + a_{-2})]&\\
+ [3,\ 4]\, a_1 a_4 + [3,\ -1]\, a_{-1} a_{-4} + [3]\, 2a_2 a_{-1},&\\
a_0 a_{-3} = [-3,\ -2]\, a_1 a_{-2} + [-3,\ -1]\, a_{-1} a_2 + (-3)\, 2a_1 a_0&\\
+ \frac{a_0}{a'}[A'_{-3} a_0^2 + B'_{-3} a_0(2a_2 + a_{-2})]&\\
+ [-3,\ 1]\, a_1 a_4 + [-3,\ -4]\, a_{-1} a_{-4} + (-3)\, 2a_2 a_{-1},&
\end{aligned}
\right\} \quad (7)
$$

$$\left.\begin{aligned} a_0 a_5 &= [5,\,1]\,a_1 a_{-4} + [5,\,4]\,a_4 a_{-1} + [5,\,2]\,a_2 a_{-3} \\ &\qquad + [5,\,3]\,a_3 a_{-2} + [5](2a_3 a_0 + 2a_2 a_1) + \frac{a_0}{a'} A_5 . 3a_0 a_2, \\ a_0 a_{-5} &= [-5,\,-4]\,a_1 a_{-4} + [-5,\,-1]\,a_4 a_{-1} + [-5,\,-3]\,a_2 a_{-3} \\ &\qquad + [-5,\,-2]\,a_3 a_{-2} + (-5)(2a_3 a_0 + 2a_2 a_1) + \frac{a_0}{a'} A'_{-5} . 3a_0 a_2 . \end{aligned}\right\} \quad (8)$$

The usual method is to solve the equations (6) for a first approximation of a_1, a_{-1} by neglecting a_3, a_{-3}, obtain a_3, a_{-3} from (7) with these values of a_1, a_{-1} and proceed then for a second approximation of a_1, a_{-1}. I prefer to do this in one process as follows.

Since a_2, a_{-2}, a_4, a_{-4} are known expansions of m in series, we can to the order given, write the equations for a_3, a_{-3} in the form

$$\left.\begin{aligned} a_3 &= \alpha a_1 + \beta a_{-1} + \gamma \frac{a_0}{a'} a_0 , \\ a_{-3} &= \alpha' a_1 + \beta' a_{-1} + \gamma' \frac{a_0}{a'} a_0 , \end{aligned}\right\} \quad (9)$$

where α, β, γ, α', β', γ' are known functions of m. Substituting these values of a_3, a_{-3} in equations (6) for a_1, a_{-1}, we obtain linear equations of the form

$$\left.\begin{aligned} a_1 . \kappa + a_{-1} . \lambda &= \mu . \frac{a_0}{a'} a_0 , \\ a_1 . \kappa' + a_{-1} . \lambda' &= \mu' . \frac{a_0}{a'} a_0 , \end{aligned}\right\} \quad (10)$$

where κ, λ, μ, κ', λ', μ' are known functions of m. This method of procedure has the advantage of giving us the whole value of the functions of m used when dealing numerically with the equations; while in finding their expansions algebraically, we lose much less than by the usual method. Also, if we wish to proceed to a higher approximation, by using the equations always in the forms (9) and (10), we shall merely get terms in (9) independent of a_1, a_{-1}, and consequently in (10) terms added only to the right-hand sides of the equations. Then the coefficients of a_1, a_{-1} in (10) will remain unchanged, and the small changes in μ, μ' can be easily obtained. This will not be found necessary here, as the results obtained up to the order indicated above are sufficiently accurate.

Expressed in the form (9), equations (7) become

$$\left.\begin{aligned} a_3 &= m^2\left(\frac{51}{2^7}+\frac{3371}{3^2.\,2^9}m+\frac{13451}{3^3.\,2^{10}}m^2\right)a_1 - m^2\left(\frac{15}{2^7}+\frac{167}{2^9}m+\frac{607}{3.\,2^9}m^2\right)a_{-1} \\ &\qquad + m^2\left(\frac{5}{2^7}+\frac{45}{2^9}m+\frac{545}{3.\,2^{11}}m^2-\frac{781}{3^2.\,2^{12}}m^3\right)\frac{a_0}{a'}\,a_0, \\ a_{-3} &= m^2\left(-\frac{25}{2^7}+\frac{103}{3^2.2^9}m+\frac{2659}{3^3.2^{10}}m^2\right)a_1 - m^2\left(\frac{3}{2^7}+\frac{19}{2^9}m+\frac{215}{3.\,2^9}m^2\right)a_{-1} \\ &\qquad - m_2\left(\frac{55}{2^7}+\frac{175}{2^9}m-\frac{229}{3.\,2^{11}}m^2-\frac{8455}{3^2.\,2^{12}}m^3\right)\frac{a_0}{a'}\,a_0, \end{aligned}\right\}(11)$$

and therefore from equation (6) we can obtain

$$\left.\begin{aligned} &a_1\left(3+\frac{29}{16}m^2+\frac{7}{2}m^3-\frac{1163}{3.\,2^{10}}m^4-\frac{209893}{3^2.\,2^{12}}m^5\right) \\ &\qquad + a_{-1}\left(1+\frac{61}{16}m^2+\frac{13}{3}m^3-\frac{68563}{3^2.\,2^{10}}m^4-\frac{338477}{3^3.2^{12}}m^5\right) \\ &= -\frac{a_0}{a'}\cdot a_0\left(\frac{3}{2}m^2+\frac{3}{4}m^3-\frac{12795}{2^{10}}m^4-\frac{96643}{2^{12}}m^5-\frac{752323}{3.\,2^{15}}m^6\right), \\ &a_1\left(3+2m+\frac{15}{16}m^2+\frac{23}{8}m^3+\frac{20645}{3.\,2^{10}}m^4+\frac{362467}{3^2.\,2^{12}}m^5\right) \\ &\qquad + a_{-1}\left(1+2m-\frac{29}{16}m^2-\frac{17}{24}m^3-\frac{55379}{3^2.\,2^{10}}m^4-\frac{891109}{3^3.\,2^{12}}m^5\right) \\ &= \frac{a_0}{a'}\cdot a_0\left(\frac{3}{8}m^2-\frac{3165}{2^{10}}m^4-\frac{21333}{2^{12}}m^5-\frac{1300401}{3.\,2^{15}}m^6\right). \end{aligned}\right\}(11a)$$

Subtract the first of these equations from the second, the resulting equation is then divisible by m; also, multiply the second equation by 3 and subtract from the first. We have then two equations from which a_1, a_{-1} must be found. Performing these processes, we get

$$\left.\begin{aligned} \frac{a_1}{a_0} &= \frac{a_0}{a'}\cdot\frac{1}{m\tau}\left[-\frac{15}{32}m^2-\frac{15}{16}m^3+\frac{123}{32}m^4+\frac{13975}{2^{10}}m^5+\frac{1709047}{3.\,2^{15}}m^6\right], \\ \frac{a_{-1}}{a_0} &= \frac{a_0}{a'}\cdot\frac{1}{m\tau}\left[+\frac{45}{32}m^2+\frac{21}{16}m^3-\frac{2763}{2^8}m^4-\frac{13449}{2^9}m^5-\frac{2979511}{3.\,2^{15}}m^6\right], \end{aligned}\right\}(12)$$

where $m\tau$, the common divisor, is given by

$$m\tau = m - 4m^2 - \frac{37}{8}m^3 - \frac{17}{6}m^4 - \frac{89963}{3^2.\,2^{10}}m^5.$$

We can of course divide out by m. It has been kept in here to show in what manner a transformation of the form

$$m = \frac{m'}{1 + am'},$$

if indicated by theory, must be made.* We should get different expressions for the coefficients a_1 and a_{-1}, kept still in the *form* (12) according as the factor m had been divided out or not.

Using the values (12), after dividing out by the factor m, since it can now be any time replaced if necessary, equations (11) give

$$\frac{a_3}{a_0} = m^2 . \frac{a_0}{a'} \Big[\frac{5}{2^7} + \frac{45}{2^9} m + \frac{545}{3 . 2^{11}} m^2 - \frac{781}{3^2 . 2^{13}} m^3 - \frac{m}{\tau} \Big(\frac{45}{2^7} + \frac{8165}{3 . 2^{11}} m - \frac{66251}{3^2 . 2^{13}} m^2 \Big) \Big],$$

$$\frac{a_{-3}}{a_0} = m^2 . \frac{a_0}{a'} \Big[- \frac{55}{2^7} - \frac{175}{2^9} m + \frac{229}{3 . 2^{11}} m^2 + \frac{8455}{3^2 . 2^{13}} m^3 + \frac{m}{\tau} \Big(\frac{15}{2^8} + \frac{551}{3 . 2^{11}} m - \frac{7459}{3^2 . 2^{10}} m^2 \Big) \Big],$$

and thence from equations (8),

$$\frac{a_5}{a_0} = \frac{a_0}{a'} m^4 \Big[\frac{105}{2^{11}} + \frac{797}{2^{13}} m - \frac{2655}{2^{13}} \cdot \frac{m}{\tau} \Big],$$

$$\frac{a_{-5}}{a_0} = \frac{a_0}{a'} m^4 \Big[- \frac{65}{2^{11}} - \frac{523}{3 . 2^{11}} m - \frac{75}{2^{13}} \cdot \frac{m}{\tau} \Big].$$

This method has the advantage also of giving numerical results along with the algebraical expansions, and thus we can obtain some idea of the errors which are produced by the portions of the series neglected when we expand the various functions used, in powers of m. In working out the numerical results by this method, it appeared that the principal parts of the errors in a_1, a_{-1} were due to the neglect of the higher powers of m in solving the linear equations for a_1 and a_{-1}, and that even this part was not very great. For example, the value of τ found was

$$1 - 4m - \frac{37}{8} m^2 - \frac{17}{6} m^3 - \frac{89963}{3^2 . 2^{10}} m^4,$$

giving $\tau = .6443757.$

The value found by the numerical process to the same order is

$$\tau = .6444540,$$

a difference of .0000783.

---

*See Monthly Not. R. A. S., Vol. LII, No. 2.

The term $$-\frac{89963}{3^2 . 2^{10}} m^4 = -.0004171,$$

so that the neglected portion is less than one-fifth of the last term in τ calculated.

The expressions given above, when transformed, agree with those given by Delaunay as far as the order $m^4 . a_0/a'$. The multiplier of $m^5 . a_0/a'$ differs in all these coefficients by a small amount, causing, however, only a small difference in the numerical values of the coefficients of a few ten-thousandths of a second.

V.

The Determination of the Portions of the Coefficients depending on Powers of $1/a'$ higher than the first.

If now we wish to obtain the portions of the coefficients depending on $(a_0/a')^2$ and $(a_0/a')^3$, very little extra labor is necessary. In the coefficients with even suffixes only even powers, and in those with odd suffixes, only odd powers of this ratio occur. Let δa_r denote the portion to be added to a_r depending on the square or cube of this ratio according as r is even or odd. The even coefficients will be obtained below as far as the order $m^3 (a_0/a')^2$, and the odd ones as far as the order $m^2 (a_0/a')^3$. As δa_0 is of the order $m^2 (a_0/a')^2$, we have to the required degree of accuracy from equation (5), after obtaining the necessary terms in L, M,

$$\left.\begin{aligned}
a_0 \delta a_2 &= [2,\ 1]\, a_1 a_{-1} + [2,\ 3]\, a_3 a_1 + [2,\ -1]\, a_{-1} a_{-3} \\
&\qquad + \frac{a_0}{a'} [3 A_2 a_0 a_{-1} + B_2 a_0 (2a_1 + a_{-1})] + \left(\frac{a_0}{a'}\right)^2 D_2 a_0^2, \\
a_0 \delta a_{-2} &= [-2,\ -1]\, a_1 a_{-1} + [-2,\ 1]\, a_3 a_1 + [-2,\ -3]\, a_{-1} a_{-3} \\
&\qquad + \frac{a_0}{a'} [3 A'_{-2} a_0 a_{-1} + B'_{-2} a_0 (2a_1 + a_{-1})] + \left(\frac{a_0}{a'}\right)^2 D'_{-2} a_0^2, \\
a_0 \delta a_4 &= [4,\ 3]\, a_3 a_{-1} + [4,\ 1]\, a_1 a_{-3} + \frac{a_0}{a'} . 3 A_4 a_0 a_1 + \left(\frac{a_0}{a'}\right)^2 C_4 a_0^2, \\
a_0 \delta a_{-4} &= [-4,\ -1]\, a_3 a_{-1} + [-4,\ -3]\, a_1 a_{-3} + \frac{a_0}{a'} . 3 A'_{-4} a_0 a_1 + \left(\frac{a_0}{a'}\right)^2 C'_{-4} a_0^2, \\
a_0 \delta a_3 &= \left(\frac{a_0}{a'}\right)^3 G_3 a_0^2, \qquad a_0 \delta a_{-3} = \left(\frac{a_0}{a'}\right)^3 G'_{-3} a_0^2, \\
a_0 \delta a_5 &= \left(\frac{a_0}{a'}\right)^3 F_5 a_0^2, \qquad a_0 \delta a_{-5} = \left(\frac{a_0}{a'}\right)^3 F'_{-5} a_0^2,
\end{aligned}\right\} . \quad (13)$$

For a_1 and a_{-1} we must use the equations (6). Differentiating these and remembering the orders to which our quantities are carried, we get

$$a_0\delta a_1 P + a_0\delta a_{-1}P' + 7a_1\delta a_2 + a_{-1}\delta a_{-2} = -\,m^2(\delta L_1 + 2m\delta M_1),$$
$$a_0\delta a_1 Q + a_0\delta a_{-1}Q' + 5a_1\delta a_2 - a_{-1}\delta a_{-2} = \quad m^2\delta M_1,$$

where P, Q, P', Q' are the same functions of m which we had previously as multipliers of a_1 and a_{-1} in their respective equations. Treat these in the same manner as we did the corresponding ones for a_1, a_{-1}, i.e. subtract the first equation from the second and also multiply the second equation by 3 and add to the first, we get

$$a_0\delta a_1(Q - P) + a_0\delta a_{-1}(Q' - P') = m^2[\delta L_1 + \delta M_1(1+2m)] + 2a_1\delta a_2 + 2a_{-1}\delta a_{-2},$$
$$a_0\delta a_1(3Q + P) + a_0\delta a_{-1}(3Q' + P') = m^2[-\delta L_1 + \delta M_1(3-2m)] - 22a_1\delta a_2 + 2a_{-1}\delta a_{-2},$$

corresponding to the equations (11a), the coefficients on the left-hand side being the same. Also

$$\delta L_1 = a_0\left(\frac{a_0}{a'}\right)^2\left[\frac{25}{16}\,(3a_{-1} + a_1) + \frac{45}{32}\,2\,(a_{-1} + a_1) + \frac{45}{32}\,a_0\right],$$
$$\delta M_1 = a_0\left(\frac{a_0}{a'}\right)^2\left[\frac{5}{8}\,(3a_{-1} + a_1) + \frac{15}{64}\,a_0\right].$$

Working out these and equations (13), and retaining the factor $\frac{1}{\tau}$ wherever it occurs, we obtain the following series of values:

$$\frac{\delta a_2}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[\frac{5}{2^6} + \frac{5}{2^4}\,m + \frac{45.67}{2^{12}}\cdot\frac{m}{\tau} + \frac{45}{2^{11}}\,(15 + 49m)\,\frac{1}{\tau^2}\right],$$
$$\frac{\delta a_{-2}}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[-\frac{45}{2^6} - \frac{5}{3.2^3}\,m - \frac{15.1253}{2^{12}}\cdot\frac{m}{\tau} + \frac{45}{2^{11}}\,(5 + 3m)\,\frac{1}{\tau^2}\right],$$
$$\frac{\delta a_4}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[\frac{7}{2^9} + \frac{7}{3.5.2^4}\,m - \frac{15.25}{2^{12}}\cdot\frac{m}{\tau}\right],$$
$$\frac{\delta a_{-4}}{a_0} = m^2\left(\frac{a_0}{a'}\right)^2\left[-\frac{7.17}{2^9} - \frac{21}{5.2^5}\,m + \frac{15.45}{2^{12}}\cdot\frac{m}{\tau}\right],$$
$$\frac{\delta a_1}{a_0} = \quad \left(\frac{a_0}{a'}\right)^3\left[-\frac{105}{2^8}\,m - \frac{210}{2^8}\,m^2 - \frac{1125}{2^9}\cdot\frac{m^2}{\tau}\right]\frac{1}{\tau},$$
$$\frac{\delta a_{-1}}{a_0} = \quad \left(\frac{a_0}{a'}\right)^3\left[\frac{315}{2^8}\,m + \frac{270}{2^8}\,m^2 + \frac{3375}{2^9}\cdot\frac{m^2}{\tau}\right]\frac{1}{\tau},$$
$$\frac{\delta a_3}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\cdot\frac{35}{3.2^{11}},$$
$$\frac{\delta a_{-3}}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\left(-\frac{35.43}{3.2^{11}}\right),$$
$$\frac{\delta a_5}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\cdot\frac{21}{5.2^{11}},$$
$$\frac{\delta a_{-5}}{a_0} = m^2\cdot\left(\frac{a_0}{a'}\right)^3\left(-\frac{1533}{5.2^{11}}\right).$$

When these are compared with Delaunay's expressions they all agree as far as the order to which he has carried them.

The results obtained by using numerical values from the outset are:

$$\frac{a_1}{a_0} = -.0641700\ \frac{a_0}{a'} - .086\ \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-1}}{a_0} = +.1789909\ \frac{a_0}{a'} + .240\ \left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_2}{a_0} = -.0000589\ \frac{a_0}{a'} + .0001\left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-2}}{a_0} = -.0029370\ \frac{a_0}{a'} - .0016\left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a^5}{a_0} = +.00000047\frac{a_0}{a'} + .0000\left(\frac{a_0}{a'}\right)^3,$$

$$\frac{a_{-5}}{a_0} = -.00000184\frac{a_0}{a'} - .0010\left(\frac{a_0}{a'}\right)^3,$$

$$\frac{\delta a_2}{a_0} = +.007284\left(\frac{a_0}{a'}\right)^2, \qquad \frac{\delta a_{-2}}{a_0} = -.007432\left(\frac{a_0}{a'}\right)^2;$$

$$\frac{\delta a_4}{a_0} = +.000060\left(\frac{a_0}{a'}\right)^2, \qquad \frac{\delta a_{-4}}{a_0} = -.001428\left(\frac{a_0}{a'}\right)^2.$$

The quantity δa_0 must now be obtained. For this purpose I use the equation

$$\left(D^2 + 2mD - \frac{\varkappa}{(us)^{\frac{3}{2}}}\right)u = -\frac{3}{2}\,m^2(u+s) - m^2 \cdot \frac{2}{n'^2} \cdot \frac{d\Omega_1}{ds}.$$

Substituting in this

$$u = \Sigma a_{i-1}\zeta^i, \quad s = \Sigma a_{-i-1}\zeta^i,$$

and taking out the coefficient of ζ we get with the former notation,

$$\varkappa\left(\frac{u}{(us)^{\frac{3}{2}}}\right)_1 = \left(1+2m+\frac{3}{2}\,m^2\right)a + \frac{3}{2}\,m^2a_{-2} + \frac{m^2}{a'}\left[\frac{15}{8}\,(s^2)_1 + \frac{3}{8}\,(u^2+2us)_1\right]$$
$$+ \frac{m^2}{a'^2}\left[\frac{35}{16}\,(s^3)_1 + \frac{5}{16}\,(u^3)_1 + \frac{15}{16}\,(us^2)_1 + \frac{9}{16}\,(u^2s)_1\right] + \dots.$$

When the parallactic terms are neglected Hill finds that

$$a_0 = a\left[\frac{J(1+m)^2}{H}\right]^{\frac{1}{3}},$$

where

$$\mu = n^2a^3, \qquad J = a_0^2\left(\frac{u}{(us)^{\frac{3}{2}}}\right)_1,$$

and $a_0 H$ is the value of the right-hand side of the equation. Let δa_0, δJ, δH be the corresponding quantities due to the parallactic terms. Then since

$$J + \delta J, \ H + \delta H$$

are defined in the same way as J, H, we have

$$a_0 + \delta a_0 = a\left[\frac{(J+\delta J)(1+m)^2}{H+\delta H}\right]^{\frac{1}{3}}.$$

Neglecting powers of $\frac{a_0}{a'}$ above the second, we get

$$\frac{\delta a_0}{a_0} = \frac{1}{3}\cdot\frac{\delta J}{J} - \frac{1}{3}\cdot\frac{\delta H}{H}.$$

We shall obtain δa_0 to the order $m^3.(a_0/a')^2$. For this

$$\delta H = m^2.\frac{a_0}{a'}\left[\frac{3}{8}.2\frac{a_{-1}}{a_0} + \frac{3}{4}\frac{(a_1+a_{-1})}{a_0}\right] + m^2.\frac{a_0^2}{a'^2}\cdot\frac{9}{16}$$

$$= \left[\frac{9}{16}m^2 + \frac{225}{2^7}\cdot\frac{m^3}{\tau}\right]\left(\frac{a_0}{a'}\right)^2.$$

$J+\delta J$ is the coefficient of ζ in $\frac{a_0^2 u}{(us)^{\frac{3}{2}}}$ or of ζ^0 in $\frac{(\Sigma a_i\zeta^i)\, a_0^2}{(\Sigma a_j a_{-i+j}\zeta^i)^{\frac{3}{2}}}$; whence we get

$$\delta J = -\frac{3}{2}\cdot\frac{a_1^2 + a_{-1}^2}{a_0^2} + \frac{9}{4}\cdot\frac{(a_1+a_{-1})^2}{a_0^2}$$

$$= -\frac{m^2}{\tau^2}\cdot\frac{675}{2^9}(1+4m)\left(\frac{a_0}{a'}\right)^2.$$

Also

$$H = 1 + 2m + \frac{3}{2}m^2 + \frac{3}{2}m^2\frac{a_{-2}}{a_0},$$

$$J = 1 + \frac{21}{2^8}m^4 + \dots,$$

$$a_0 \doteq a\left(1 - \frac{m^2}{6} - \dots\right),$$

we therefore have

$$\frac{\delta a_0}{a_0} = -\left(\frac{a}{a'}\right)^2 m^2\left\{\frac{225}{512}\cdot\frac{1+4m}{\tau^2} + \frac{75}{128}\cdot\frac{m}{\tau} + \frac{3}{16}(1-2m)\right\},$$

and numerically,

$$\frac{\delta a_0}{a_0} = -\left(\frac{a_0}{a'}\right)^2.00965.$$

VI.

Transformation to Polar Coordinates.

The inequalities will now be expressed in polar coordinates. For this purpose we have, if V be the true longitude of the moon,

$$
\begin{aligned}
r\cos(V-nt) &= \frac{1}{2}\Sigma a_i(\zeta^i+\zeta^{-i}) \\
&= a_0\left\{1+\frac{1}{2}\Sigma\frac{a_i}{a_0}(\zeta^i+\zeta^{-i})\right\} \\
&= a_0\left(1+\frac{1}{2}\Sigma A_i\zeta^i\right), \\
r\sin(V-nt) &= \frac{1}{2\sqrt{-1}}\Sigma a_i(\zeta^i-\zeta^{-i}) \\
&= a_0\frac{1}{2\sqrt{-1}}\Sigma B_i\zeta^i,
\end{aligned}
$$

where $A_i.a_0=a_i+a_{-i}$ and $B_i.a_0=b_i-b_{-i}$, and consequently $A_i=A_{-i}$, $B_i=-B_{-i}$. And since

$$\tan\theta=\theta-\frac{\theta^3}{3}+\ldots\ldots,$$

$$
\begin{aligned}
V-nt = \frac{1}{2\sqrt{-1}}[\Sigma B_i\zeta^i]\Big[1-\frac{1}{2}\Sigma A_i\zeta^i+\frac{1}{4}(\Sigma A_i\zeta^i)^2-\ldots\ldots\Big] \\
+\frac{1}{24\sqrt{-1}}[\Sigma B_i\zeta^i]^3[1-\ldots\ldots]^3+\ldots\ldots
\end{aligned}
$$

The coefficient of $\sin D$ or $(\zeta^1-\zeta^{-1})/2\sqrt{-1}$ is therefore

$$B_1+\frac{1}{2}(B_1A_2-A_1B_2)+\frac{1}{2}(B_2A_3-A_2B_3)+\frac{1}{2}B_1(A_2^2-B_2^2)+\ldots\ldots,$$

or $$\frac{a_1-a_{-1}}{a_0}+\frac{a_1a_{-2}-a_2a_{-1}+a_2a_{-3}-a_3a_{-2}}{a_0^2}+\frac{2(a_1-a_{-1})a_2a_{-2}}{a_0^3}+\ldots\ldots,$$

and similarly for the other coefficients. Performing these operations we get the following expressions, omitting those terms dependent on m only.

In longitude,

$$
\begin{aligned}
\Bigg[-\frac{\frac{15}{8}m+\frac{9}{4}m^2-\frac{1951}{128}m^3-\frac{41585}{1024}m^4-\frac{2096751}{49152}m_5}{\underbrace{1-4m-\frac{37}{8}m^2-\frac{17}{6}m^3-\frac{89963}{9216}m^4(=\tau)}_{-128''.0545}}-\underset{-0''.0008}{\frac{35}{1024}m^4}-\underset{-0''.0002}{\frac{1345}{12288}m^5} \\
-\frac{a^2}{a'^2}\left\{\underset{-0''.0006}{\frac{105}{64}\cdot\frac{m}{\tau}}+\underset{-0''.0001}{\frac{15}{8}\frac{m^2}{\tau}}+\underset{-0''.0004}{\frac{1125}{128}\cdot\frac{m^2}{\tau^2}}\right\}\Bigg]\frac{a}{a'}\sin D
\end{aligned}
$$

$$+\left[\underset{0''.0078}{\frac{5m^2}{32}(5+8m)}+\underset{+0''.0276}{\frac{225}{1024}m^2\frac{(5+11m)}{\tau^2}}+\underset{+0''0060}{\frac{15.747}{2048}\cdot\frac{m^3}{\tau}}\right]\frac{a^2}{a'^2}\sin 2D$$

$$+\left[\underset{1''.6172}{\frac{15}{32}m^2}+\underset{+0''.1199}{\frac{55}{128}m^3}-\underset{-0''.0006}{\frac{41}{1536}m^4}-\underset{-0''.0005}{\frac{2309}{9216}m^5}-\underset{-0''.0872}{\frac{1}{\tau}\left(\frac{255}{128}m^3+\frac{7543}{1536}m^4-\frac{134525}{8192}m^5\right)}\right.$$

$$\left.+\underset{+0''.0000}{\frac{a^2}{a'^2}\cdot\frac{385}{1536}m^2}\right]\frac{a}{a'}\sin 3D$$

$$+\left[\underset{0''.0021}{\frac{63}{256}m^2}+\underset{+0''.0001}{\frac{77}{480}m^3}-\underset{-0''.0008}{\frac{1725}{2048}\cdot\frac{m^3}{\tau}}\right]\frac{a^2}{a'^2}\sin 4D$$

$$+\left[\underset{0''.0132}{\frac{75}{128}m^4}+\underset{+0''.0025}{\frac{2797}{2048}m^5}-\underset{-0''.0064}{\frac{18415}{8192}\cdot\frac{m^5}{\tau}}+\underset{+0''.0000}{\frac{a^2}{a'^2}\cdot\frac{777}{5120}m^2}\right]\frac{a}{a'}\sin 5D.$$

In parallax,

$$\frac{1}{a}\left[\underset{-1''.0100}{-\left(\frac{15}{16}m+\frac{3}{8}m^2-\frac{1869}{256}\right)\frac{1}{\tau}}\right]\frac{a}{a'}\cos D$$

$$+\frac{1}{a}\left[\underset{0''.0323}{\frac{25}{64}m^2}+\underset{+0''.0012}{\frac{65}{256}m^3}-\underset{-0''.0139}{\frac{459}{256}\cdot\frac{m^3}{\tau}}\right]\frac{1}{a'}\cos 3D.$$

None of the other parts produce in parallax, coefficients so great as one thousandth of a second.

The numerical values of these coefficients are obtained by using

$$m=.0808489338,$$

$$\frac{a}{a'}=.00255878.$$

If we use the numerical value of m from the outset, we obtain the more accurate series of coefficients:

In longitude,

$$-128''.070\sin D+0''.039\sin 2D+0''.750\sin 3D+0''.001\sin 4D+0''.008\sin 5D,$$

and in parallax

$$-1''.001\cos D+0''.008\cos 3D.$$

For the discussion and comparison of these results with those obtained by Delaunay, see Monthly Notices of Royal Astron. Soc., Vol. LII, No. 2.

NOVEMBER, 1891.

On the Curves which are Self-reciprocal in a Linear Nulsystem, and their Configurations in Space.

A paper read before the New York Mathematical Society at the Meeting of Nov. 7, 1891.

BY CHARLES PROTEUS STEINMETZ, *Yonkers, N. Y.*

---

§1.—INTRODUCTION.

Every conic curve defines in the plane a polar system; that is, a linear involutory reciprocal correspondence between the points and the lines of the plane.

In a similar way the simplest rational curve in space, the twisted cubic, defines a reciprocal correspondence also, whereby the points and the planes in space are related to each other so that to every point $\mathfrak{x}$ corresponds the connecting plane ξ of the osculating points of the three osculating planes drawn from $\mathfrak{x}$ onto the cubic curve, while inversely to every plane ξ corresponds the point of intersection $\mathfrak{x}$ of the osculating planes of the cubic curve, drawn in the 3 points of intersection of the plane ξ with the cubic curve. Hence to the points of the cubic curve correspond its osculating planes, and inversely, so that this cubic curve is self-reciprocal in this polar system in space.

Now, the plane ξ always passing through its corresponding point $\mathfrak{x}$, this linear involutory reciprocal correspondence is coincident; that means, corresponding elements coincide. Hence this polar system in space is a nulsystem, and ξ the nulplane of the nulpol $\mathfrak{x}$.

The nulsystem defined by the twisted cubic is the linear nulsystem.

But reversing the reasoning, we find that to every polar system in the plane belongs a conic curve—which indeed may be imaginary—while on the other hand, a nulsystem in space does not define a curve as self-reciprocal element, but an infinite number of curves, amongst which there are straight lines, cubics and higher algebraic curves, and transcendental curves.

With these curves, which are self-reciprocals in a linear nulsystem in space, and with the configurations of these curves, we shall deal in the following, especially with cubic curves.

A curve C will be called self-reciprocal in the linear nulsystem in space, if to every point $\mathfrak{x}$ of the curve C as nulpol, corresponds the osculating plane ξ of this point as nulplane. Then, from the properties of the nulsystem follows, that the osculating points of all the osculating planes of the curve C which pass through a given point $\mathfrak{x}$, lie upon a plane ξ, which passes through point $\mathfrak{x}$ as its nulpol, and that the osculating planes in the points of intersection of the curve C with any plane ξ intersect in a point $\mathfrak{x}$, which lies upon ξ as its nulplane.

A surface S will be called self-reciprocal in the linear nulsystem, if to every point $\mathfrak{x}$ of the surface S corresponds a tangent plane ξ of this surface S as nulplane, and inversely.

But in general the nulplane ξ is not the tangent plane of S in point $\mathfrak{x}$, but in another point $\mathfrak{y}$. Only a finite number of points $\mathfrak{z}$ of the surface S can exist, which have their own tangent plane as nulplane, and shall be called the singular points of the surface S.

Surfaces of that kind, that every tangent plane is nulplane of its own tangent point, cannot exist. For in such a surface all the tangent planes through a point $\mathfrak{x}$ outside of S would touch S in a plane ξ passing through $\mathfrak{x}$, and their tangent-points therefore lie on a plane curve of n^{th} order. But the tangent points of the planes through $\mathfrak{x}$ lie on a curve of $n(n-1)^{\text{th}}$ order, and therefore

$$n = n(n-1);$$

hence

$$n = 0, \text{ the point;}$$

$$n = 2, \text{ the quadric surface.}$$

But the quadric surface defines a polar system where the plane ξ cannot pass through $\mathfrak{x}$, hence never can be a nulsystem.

Plane curves cannot be self-reciprocals in the linear nulsystem except when being straight lines.

For in a plane self-reciprocal curve to every point of the curve its plane would have to correspond as nulplane, what is impossible.

Self-reciprocal straight lines exist as a tridimensional system, the "self-conjugate rays" of the nulsystem, and are grouped in plane pencils.

Self-reciprocal conics cannot exist.

Self-reciprocal cubics form a 7-dimensional system, and are grouped into pencils also on conic surfaces, which we shall treat in the following.

Self-reciprocal quartics exist also. They are rational quartics with two stationary tangents.

§2.—*Self-reciprocal Algebraic Curves in the Nulsystem.*

Besides cubics and quartics, there exist still an infinite number of self-reciprocal curves in the linear nulsystem.

For to any point $\mathfrak{a}_1$ corresponds a nulplane α_1. Now assume a point $\mathfrak{a}_2$ on α_1, infinitely near to $\mathfrak{a}_1$. To $\mathfrak{a}_2$ corresponds the nulplane α_2 which passes through $\mathfrak{a}_1$ and $\mathfrak{a}_2$. Assume $\mathfrak{a}_3$ on α_2 infinitely near to $\mathfrak{a}_2$ and produce its nulplane α_3, etc.

Then the points $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \ldots . \mathfrak{a}_\infty$ have as nulplanes

$$\alpha_1, \alpha_2 = |\mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3|, \ \alpha_3 = |\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4|, \ldots . \alpha_\infty,$$

while the lines

$$t_1 = |\mathfrak{a}_1\mathfrak{a}_2|, \quad t_2 = |\mathfrak{a}_2\mathfrak{a}_3|, \ldots . t_\infty$$

are self-conjugate rays. Hence the curve consisting of the points $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \ldots . \infty$, having as osculating planes their nulplanes $\alpha_1, \alpha_2 \ldots . \alpha_\infty$, and as tangents the self-conjugate rays $t_1, t_2 \ldots . t_\infty$, must therefore be a self-reciprocal curve.

Every one of these self-reciprocal curves defines the nulsystem.

All the self-reciprocal curves of the linear nulsystem have the essential property that—

1). The osculating points of all the osculating planes drawn through any point $\mathfrak{x}$, taken at random in space, lie on a plane ξ passing through $\mathfrak{x}$.

2). The osculating planes in all the points of intersection of the curve with a plane ξ, taken at random, intersect each other in a point $\mathfrak{x}$ of this plane.

For algebraic self-reciprocal curves hold the conditions—

1). The class m of the curve, that is, the number of osculating planes through any point in space, is equal to the order n of the curve; that is, the number of points of intersection of the curve with any plane in space.

2). Stationary osculating planes cannot exist (except in stationary tangents, which represent each two stationary osculating planes), for a stationary osculating plane would have two nulpols infinitely near with each other, its two osculating points.

3). The same, cuspidal points cannot exist.

4). Simple double points and double osculating planes cannot exist.

But a combination of double point and double osculating plane can exist; that is, a double point whose two osculating planes coincide in one double osculating plane.

5). To determine the rank of the curve; that is, the order of the associate developable surface or the number of tangents intersecting a straight line g, we consider the involution of n^{th} order produced on this line g.

Let g' be the conjugate line to g in the nulsystem. Then to any point $\mathfrak{p}'$ of g' corresponds a nulplane π' which passes through g. This plane π' intersects the curve C in n points $\mathfrak{x}_i$, and the n lines $|\mathfrak{p}'\mathfrak{x}_i|$ cut g in n points of an involution.

This involution contains $2(n-1)$ double points.

In each of those $2(n-1)$ points coincide two lines $|\mathfrak{p}'\mathfrak{x}_i|$. Now these lines $|\mathfrak{p}'\mathfrak{x}_i|$ are self-conjugate rays because of intersecting two conjugate rays g and g'.

Hence, when two lines $|\mathfrak{p}'\mathfrak{x}_i|$ coincide, this line must contain two points $\mathfrak{x}_i$ of the curve, infinitely near with each other; that is, it must be a tangent.

Hence $2(n-1)$ tangents of C intersect g, or

"The rank of the curve C is $r=2(n-1)$."

6). Number of chords through a point $\mathfrak{x}$, or number of apparent nodes of the curve.

Projecting the twisted curve C from a point $\mathfrak{p}$ upon a plane ε, we derive in this plane, as the image of curve C, a curve c of n^{th} order.

The points $\mathfrak{x}'$ and tangents $t'_{\mathfrak{x}}$ of this curve c are the projections of the points $\mathfrak{x}$ and tangents $t_{\mathfrak{x}}$ of the twisted curve C.

C being of the rank $2(n-1)$, c is of the class $2(n-1)$.

Hence if d is the number of (real) double points or nodes of C, s is the number of apparent double points or of chords onto C from $\mathfrak{p}$, the curve c has $(d+s)$ double points.

If $\mathfrak{p}$ is taken at random in space, but outside of the developable surface determined by C, no tangent of C passes through $\mathfrak{p}$, and C having no cuspidal points, c has no cusps either.

Hence it is

$$2(n-1)=n(n-1)-2(d+s),$$

or

$$d+s=\frac{(n-1)(n-2)}{2},$$

the number of apparent and true nodes of C.

Now let

n = the *order* of the self-reciprocal curve C, or the number of points of intersection with a plane ε.
m = the *class* of C; or the number of osculating planes through a point $\mathfrak{p}$.
r = the *rank* of C, or the number of tangents intersecting a line g.
d = the number of double points of C.
s = the number of apparent double points, or of chords through a point $\mathfrak{p}$.
v = the number of biosculating planes.
u = the number of osculating lines; that is, lines of intersection of two osculating planes, in a plane ε.
t = the number of stationary tangents.
k = the order of the nodal curve of the osculating developable.
l = the class of the bitangent developable.

The number of cuspidal points and stationary osculating planes being $= 0$.

Then we have

1). $n = m$.
2). $d = v$.
3). $k = l$. (I)
4). $r = 2(n-1)$.
5). $s + d = u + v \dfrac{(n-1)(n-2)}{2}$.

But, between these constants of the curve C,

$$n,\ m,\ r,\ d,\ s,\ v,\ u,\ t,\ k,\ l$$

exist in general the six equations

1). $m = r(r-1) - 2k - 3(n+t)$.
2). $r = m(m-1) - 2(u+v)$.
3). $n + t = 3(r-m)$.
4). $n = r(r-1) - 2l - 3(m+t)$. (II)
5). $r = n(n-1) - 2(s+d)$.
6). $m + t = 3(r-n)$.

The equations (I) and (II) combined give

a). $m = n$.
b). $r = 2(n-1)$.
c). $s + d = u + v = \dfrac{(n-1)(n-2)}{2}$,
d). $k = l = 2(n-2)(n-3)$.
e). $t = 2(n-3)$.
f). $s = u,\ d = v$.

Whence we get the genus of the self-reciprocal curve

$$p = (n-1)(n-2) - 2(d+s) = 0;$$

that is,

"*The algebraic curve, which is self-reciprocal in a linear nulsystem, is of the genus zero; that is, is a rational curve.*"

From these equations (III) we derive for the first orders of self-reciprocal curves the numerical constants:

$n = m$.	r.	$s + d = u + v$.	t.	$k = l$.
1	0			
2	2			
3	4	1	0	0
4	6	3	2	4
5	8	6	4	12
6	10	10	6	24

Hence the first self-reciprocal twisted curves are the general twisted cubic, and the rational twisted quartic with two stationary tangents.

§3.—*Self-reciprocal Straight Lines and their Configurations in the Nulsystem.*

The linear nulsystem contains a tridimensional system of self-reciprocal straight lines, which are grouped into plane pencils: the self-conjugate rays of the nulsystem.

Of their configurations the most simple is the self-reciprocal tetragon, generally called "nultetrahedron."

It consists of four self-conjugate rays—

$$|\mathfrak{a}\mathfrak{b}|,\ |\mathfrak{b}\mathfrak{c}|,\ |\mathfrak{c}\mathfrak{d}|,\ |\mathfrak{d}\mathfrak{a}|,$$

which form four edges of a tetrahedron, the planes of which are the nulplanes of its vertices, while the third pair of edges is a pair of conjugate rays—

$$|\mathfrak{a}\mathfrak{c}| \text{ conjugate } |\mathfrak{b}\mathfrak{d}|.$$

Another configuration is the *nulhexagon*.

Let its vertices be $\mathfrak{p}_1,\ \mathfrak{p}_2,\ \dots\ \mathfrak{p}_6,$

its edges, $g_{12},\ g_{23},\ \dots\ g_{61},$

its sides, $\varepsilon_{123},\ \varepsilon_{234},\ \dots\ \varepsilon_{612}.$

Then the planes $\varepsilon_{i\kappa}$ are the nulplanes of the points $\mathfrak{p}_\kappa$.

The diagonals

$$l_{14} = |\mathfrak{p}_1\mathfrak{p}_4|,\ l_{25} = |\mathfrak{p}_2\mathfrak{p}_5|,\ l_{36} = |\mathfrak{p}_3\mathfrak{p}_6|$$

are conjugate rays to the three lines of intersection of the diametrically opposite planes

$$l'_{14} = |\varepsilon_{612}\varepsilon_{345}|,\ l'_{25} = |\varepsilon_{123}\varepsilon_{456}|,\ l'_{36} = |\varepsilon_{234}\varepsilon_{561}|.$$

These six lines, $l_{14}\, l'_{14}\, l_{25}\, l'_{25}\, l_{36}\, l'_{36}$ are cut by two self-conjugate rays g_1 and g_2.

The short diagonals of the hexagon

$$p_2 = |\mathfrak{p}_1\mathfrak{p}_3|,\ p_4 = |\mathfrak{p}_3\mathfrak{p}_5|,\ p_6 = |\mathfrak{p}_5\mathfrak{p}_1|$$

are conjugate rays to the lines

$$p'_2 = |\varepsilon_1\varepsilon_3|,\ p'_4 = |\varepsilon_3\varepsilon_5|,\ p'_6 = |\varepsilon_5\varepsilon_1|.$$

Therefore $p'_2 p'_4 p'_6$ intersect in a point $\mathfrak{p}$ of the plane $[p_2 p_4 p_6]$. The same $p'_1 p'_3 p'_5$ intersect in a point $\mathfrak{p}'$ of the plane $[p_1 p_3 p_5]$.

The *nuloctagon.*

Let its vertices be

$$\mathfrak{p}_1,\ \mathfrak{p}_2,\ \ldots\ \mathfrak{p}_8,$$

its edges, $g_{12},\ g_{23},\ \ldots\ g_{81},$

its sides, $\varepsilon_1,\ \varepsilon_2,\ \ldots\ \varepsilon_8.$

Each quadruple of 4 not adjoining edges is cut by two lines,

$$g_{12},\ g_{34},\ g_{56},\ g_{78} \text{ by } l_1 \text{ and } l'_1,$$

$g_{23},\ g_{45},\ g_{67},\ g_{81}$ by l_2 and l'_2, which are conjugate rays, and these four lines

$$l_1,\ l'_1,\ l_2,\ l'_2$$

are in hyperbolic position.

Hereby to every self-reciprocal octagon is associated a ruled hyperboloid.

The *nul n-gon.*

Any quadruple of not adjoining edges is cut by two conjugate rays, and any two of these pairs determine a ruled hyperboloid, etc.

§4.—*Self-reciprocal Cubic Curves in the Nulsystem.*

Every linear nulsystem contains a 7-dimensional system of self-reciprocal twisted cubics.

Through every point $\mathfrak{p}$ passes a 5-dimensional system of self-reciprocal cubics, which in $\mathfrak{p}$ osculate the nulplane ε of $\mathfrak{p}$. Every self-conjugate ray in ε is tangent of a 4-dimensional system of this 5-dimensional system.

Every self-conjugate ray is tangent of a 5-dimensional system of self-reciprocal cubics.

Two points $\mathfrak{p}_1$ and $\mathfrak{p}_2$, with their nulplanes ε_1 and ε_2, determine a tridimensional system.

Three points $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, with their nulplanes ε_1, ε_2, ε_3, determine a pencil of self-reciprocal cubics.

The nulsystem contains a 5-dimensional system of cubic parabolas,
a 5-dimensional " " " circles,
a 6-dimensional " " " hyperbolic parabolas.

§5.—*Self-reciprocal Curves on given Surfaces.*

On any surface F given at random, there exists an infinite number of curves, C, which are self-reciprocals in a given linear nulsystem.

Through every point of F always passes one and only one C, except in certain "singular points" $\mathfrak{S}$ of F, through which passes an infinite number of C's. These singular points $\mathfrak{S}$ are those points which have their tangent plane as nulplane. In these points $\mathfrak{S}$ the surface F is touched by the surface F', which is reciprocal to F in the nulsystem. Hence these points $\mathfrak{S}$ exist only in a finite number.

The self-reciprocal curve C, which passes through a given point $\mathfrak{x}_1$ of F, is produced by the infinitesimal method in the following way:

Let ξ_1 be the nulplane of $\mathfrak{x}_1$. Then ξ_1 intersects F in a curve, which has as tangent in x_1 the tangent t_1 of C. Assume on t_1 a point x_2 infinitely near to x_1. Then the nulplane ξ_2 of $\mathfrak{x}_2$ intersects F in the next tangent t_2 of C, etc.

Let $n =$ the order of the surface F, $m =$ the class, $r =$ the rank.
Then $m =$ the order of the reciprocal F', $n =$ the class, $r =$ the rank.

The order of their curve of intersection equals the class of their double-tangent developable, $= mn$, and the points of intersection of both curves are the singular points $\mathfrak{S}$.

§6.—*Self-reciprocal Curves on Ruled and Developable Surfaces.*

All the self-reciprocal curves C on a ruled surface F intersect the generatrices of F in projective ranges of points, and the tangents of C along any generatrix g from a ruled hyperboloid of self-conjugate rays which touches the reciprocal ruled surface F' in a straight line g', reciprocal to g, and besides g' intersects F' in a curve of $2(n-1)^{\text{th}}$ order.

For the tangents t of the self-reciprocals C intersect g, the infinitely near generatrix g_1, and the conjugate ray g'.

All the self-reciprocal curves C on a given developable surface F intersect the generatrices in projective ranges of points also. The tangents of C along a given generatrix g of F form a plane pencil, with the nulpol of the tangent plane of F in g as centre. Hence infinitely near generatrices of F are cut by the self-reciprocal curves C in perspective ranges of points, and the developable surfaces of the self-reciprocal curves C intersect each other in a curve K, which is the reciprocal curve of the developable surface F.

Twisted Cubics on Ruled and Developable Surfaces.

A developable surface, containing two given twisted cubics, is determined as the locus of the common tangent planes of both.

The class of this developable is, $m = 16$. For the two twisted cubics are projected from a point in space by cones of fourth class, which have 16 generatrices in common.

Every tangent of the one twisted cubic being intersected by four tangents of the other, four tangent planes of the other pass through the tangent line of the first; that is, the curves of third order are quadruple curves of the developable surface.

The osculating curve intersects either one of the twisted cubics in 12 points, the points of intersection of the one cubic with the osculating developable of the other.

A ruled surface, containing three given twisted cubics, $C_1 C_2 C_3$, is determined as the locus of the common secants of the three cubics.

This ruled surface is of the 54th order and class. For

Assume three straight lines $l_1 l_2 l_3$ at random. All the rays g intersecting these three lines $l_1 l_2 l_3$ form a ruled hyperboloid. This intersects the cubic C_1 in 6 generatrices. Hence all the rays intersecting one cubic C_1 and two straight lines $l_2 l_3$ form a ruled surface of 6th order because of intersecting a line l_1 in 6 points.

This ruled surface of 6th order intersects C_2 in 18 points.

Hence the secants of two cubics C_1C_2 and one line l_3 form a ruled surface of 18th order because of intersecting a line l_3 in 18 points.

This ruled surface of 18th order intersects C_3 in 54 points, etc.

This ruled surface of 54th order, R, which is determined by the three self-reciprocal cubics $C_1C_2C_3$, contains $C_1C_2C_3$ as 9-ple curves. For through any point of C_1 pass 9 generatrices of R, the common generatrices of the two cubic cones, which produce C_2 and C_3.

Besides these three 9-ple cubics, the surface R contains 108 double rays. For all the chords of C_1 which intersect a line l lie on a ruled surface of 4th order. This intersects C_2 in 12 points. Hence the chords of C_1 which intersect C_2 form a ruled surface of 12th order, and this intersects C_3 in 36 points. Hence there exist 36 generatrices of R which cut C_1 twice, and therefore are double generatrices of R. The same 36 double generatrices of R intersect C_2 twice, and 36 intersect C_3 twice.

The curve of intersection of a plane with the ruled surface R is of 54th order and 54th class, contains nine 9-ple points and 108 double points. Hence it contains still:

$$\tfrac{1}{2}\{54\times 53 - 9\times(9\times 8) - 2\times 108 - 54\} = 972 \text{ double points.}$$

The ruled surface R contains a double curve of 972nd order.

On this ruled surface R of 54th order exists an infinite number of self-reciprocal curves of 27th order, three of which degenerate into the cubics $C_1C_2C_3$. No two of these reciprocal curves intersect each other. Hence the surface contains no singular points.

If the three self-reciprocal cubics $C_1C_2C_3$ have one point $\mathfrak{a}$ in common, the ruled surface R breaks up into—

1). The three quadratic cones, projecting $C_1C_2C_3$ from $\mathfrak{a}$, counted twice.

2). The osculating plane α of $C_1C_2C_3$, which as nulplane of $\mathfrak{a}$ is common to $C_1C_2C_3$, counted twice also.

3). A ruled surface of 40th order, which has $\mathfrak{a}$ as singular point. If the three self-reciprocal cubics $C_1C_2C_3$ have 3 points $\mathfrak{abc}$ in common, the ruled surface R consists of—

1). The 9 quadratic cones, projecting $C_1C_2C_3$ from $\mathfrak{abc}$, counted twice.

2). The 3 osculating planes $\alpha\beta\gamma$ in $\mathfrak{abc}$ counted twice.

3). A cubic cone K, projecting the three cubics $C_1C_2C_3$ from the point $\mathfrak{d} = (\alpha\beta\gamma)$, counted 6-fold.

This cubic cone K, into which the ruled surface R degenerates, has the points $\mathfrak{abc}$ as singular points, and contains an infinite number of twisted cubics into which the curves of 27th order of R degenerated. Its centre $\mathfrak{p} = (\alpha\beta\gamma)$ is the nulpol of the connecting plane of its 3 singular points $\pi = [\mathfrak{abc}]$. The self-reciprocal cubics then pass through 3 given points $\mathfrak{abc}$ and osculate there the three planes $\alpha\beta\gamma$. They shall be considered more particularly in §9.

§7.—*Self-reciprocal Curves on Ruled Hyperboloids.*

On every ruled hyperboloid exists an infinite number of curves which are self-reciprocals in a linear nulsystem in space.

To find the singular points of the hyperboloid, we determine its self-conjugate generatrices. For any generatrix of the hyperboloid passing through a singular point, must be a self-conjugate ray, and the singular points of the hyperboloid are therefore the points of intersection of its self-reciprocal generatrices.

Let the rays of the one system of straight lines of the hyperboloid be called *generatrices*, the rays of the other system *directrices*.

Then to all generatrices and directrices of the ruled hyperboloid H, taken at random in space, correspond in the nulsystem the generatrices and directrices of another hyperboloid H'.

To a generatrix g of H corresponds a generatrix g' of H'. g' intersects H in two points. Through each one of these two points passes a directrix d_1, viz. d_2 of H, which, because of intersecting two conjugate rays g and g', is a self-conjugate ray, and therefore common ray of both hyperboloids H and H'.

In the same way we find two self-reciprocal generatrices g_1 and g_2, which are common rays of both hyperboloids.

Hence the hyperboloid H contains 4 self-reciprocal rays, 2 generatrices, g_1, g_2, and 2 directrices, d_1, d_2, which, lying on the conjugate hyperboloid H' also, represent its curve of intersection of 4th order with H.

The 4 points of intersection,

$$(g_1d_1),\ (g_1d_2),\ (g_2d_1),\ (g_2d_2)$$

are the *singular points*, and the nulplanes of these singular points are

$$[g_1d_1],\ [g_1d_2],\ [g_2d_1],\ [g_2d_2].$$

The lines $|(g_1d_1),\ (g_2d_2)|$ and $|(g_1d_2),\ (g_2d_1)|$
are conjugate rays.

Each ruled hyperboloid H contains 4 singular points, which form a nultetrahedron.

All the self-reciprocal curves on the hyperboloid intersect any two generatrices, viz. directrices, in projective ranges of points, and the connecting lines of corresponding points form a ruled hyperboloid.

In general, no twisted cubic exists amongst the self-reciprocal curves on a ruled hyperboloid. For:

Supposed on the hyperboloid H exists a self-reciprocal cubic C. This C has two generatrices of any ruled hyperboloid through H as tangents. Hence it has the self-conjugate generatrices g_1, g_2 as tangents, and osculates them in two singular points.

But the cubic C is determined by two points with their tangents g_1, g_2 and their osculating planes, and determines an hyperboloid with g_1, g_2 as generatrices where it lies on. Only when this hyperboloid is identical with H, what in general is not the case, H contains a twisted cubic.

If a ruled hyperboloid contains a twisted cubic, it contains only one, and never more. For:

Supposed H contains besides the self-reciprocal cubic C still another one, C'.

If C and C' have different systems of generatrices of H as chords, then 2 chords, g and d', together with C and C', make up two quartic curves of intersection of H with two quadrines, which have 8 common points of intersection Now g and d' intersect C' and C in one point, each other in one point, hence C and C' must have 5 points of intersection, what is impossible.

If, on the other hand, C and C' have the same system of generatrices of H as chords, then they make up two quartic curves with two generatrices g and g'. g intersects C', g': C in two points, hence C and C' must have four points of intersection, which could only be the four singular points of H. But four points of intersection of two cubics are possible only in the case, §9, which is excluded here, because then the connecting lines of the common points of intersection cannot be self-conjugate rays.

Hence the hyperboloid can contain more than one cubic only, if these have less than four points of intersection, and that is, if g and g' intersect each other, or, what is the same, if H degenerates into a quadricone. Hence, *quadric surfaces which contain more than one self-reciprocal cubic are cones.*

In general, an hyperboloid H contains two pairs of self-conjugate rays which form a nultetrahedron, but contains no pair of conjugate rays.

If an hyperboloid H contains one pair of conjugate rays, g and g', all the rays of this system of generatrices are grouped in pairs of conjugate rays, while the other system of generatrices consists entirely of self-conjugate rays. Then this hyperboloid H and its reciprocal hyperboloid H' coincide, and H is called a self-reciprocal hyperboloid. It contains of self-reciprocal curves only the rays of the one system of generatrices.

What has been said here with regard to the ruled hyperboloid holds in the same way, with due specialization for its special case, the ruled paraboloid. Of special interest is that paraboloid which touches the plane at infinity in its nulpol and thereby contains it as singular point.

§8.—*The Quadratic Pencil of Self-Reciprocal Cubics.*

We have seen in §7 that a quadratic surface can contain more than one self-reciprocal twisted cubic only when it degenerates into a quadricone.

But not every quadricone, but only such a cone which contains the nulplane of its centre as tangent plane, contains reciprocal cubics.

All the self-reciprocal twisted cubics which pass through a point $\mathfrak{P}$ and there osculate its nulplane Π, form a 5-dimensional system.

All those self-reciprocal twisted cubics which in a point $\mathfrak{P}$ have the line P as tangent and osculate Π which passes through P, form a 4-dimensional system. They are produced from $\mathfrak{P}$ by a tridimensional system of quadricones, which have in their common generatrix P the common tangent plane Π.

Assume at random one of those cones K.

All the generatrices p of K are conjugate to the tangents q of a plane conic curve $\mathfrak{K}$ in Π. The points $\mathfrak{q}$ of this cone correspond to the tangent planes of K, and therefore lie upon them.

Let us assume one of these quadricones K. Construe in any one of its points $\mathfrak{x}$ the self-conjugate tangent line x which passes through a point $\mathfrak{y}$ of the conjugate conic $\mathfrak{K}$ in Π and at the same time intersects the next generatrix of K in $\mathfrak{x}_1$. Construe again the tangent x_1, intersecting $\mathfrak{K}$ in $\mathfrak{y}_1$, etc., so we get on K a self-reciprocal curve $\mathfrak{x}, \mathfrak{x}_1 \ldots . = C$.

The curve of intersection of the osculating developable surface of this curve C with the plane Π consists of the double ray P and of the conic $\mathfrak{K}$, is therefore of 4th order. Hence this developable, being of 4th order, intersects K in a curve of 8th order, which consists of the double ray P and the double curve C. That means, C is a twisted cubic, q. e. d.

From a point $\mathfrak{r}$ of this twisted cubic C on the generatrix p, it is projected by a quadric cone, which intersects Π in the conic X. This conic X touches the conic $\mathfrak{K}$, which corresponds to K in the point $\mathfrak{P}$, and the point of intersection of x, the tangent of C in $\mathfrak{r}$ with Π.

Projecting X from any other point $\mathfrak{r}'$ of the same generatrix p by a quadratic cone, this cone intersects K in p and in a twisted cubic C'.

Any tangent y' of this cubic C' is produced from $\mathfrak{r}'$ by a plane $|\mathfrak{r}'y'|$, which intersects Π in the same line x_0, as the plane $|\mathfrak{r}y|$ producing the tangent y of C from $\mathfrak{r}$. Hence y and y' both pass through the point of intersection $\mathfrak{q}$ of x_0 with the tangent plane of K along $|\mathfrak{P}\mathfrak{p}|$. But this point $\mathfrak{q}$ is nulpol of this tangent plane, because of y being self-conjugate, and therefore y' is self-conjugate; that is, all the tangents of C' are self-conjugate rays, and C' therefore a self-reciprocal curve:

"*On the quadricone* K *exists an infinite number of self-reciprocal twisted cubics, which, from their points of intersection with any generatrix* p *of* K *are projected upon the nulplane* Π *of the centre* $\mathfrak{P}$ *of* K *by one and the same conic* X. *In this way to every generatrix* p *of* K *corresponds a conic* X *in* Π."

In consequence hereof,

"*All the twisted cubics intersect the generatrices of the quadricone* K *in perspective ranges of points, and their tangents produce projective pencils of rays in the tangent plane of* K."

For the lines $|\mathfrak{r}\mathfrak{y}|$, $\mathfrak{r}$ being point of intersection with the one, $\mathfrak{y}$ with the other generatrix, pass through one and the same point of Π, the point of intersection of the projection-conics X and Y. All the tangents of C along a generatrix p intersect in the nulpol of π on Π.

Any two self-reciprocal twisted cubics of the quadratic pencil have, besides the centre $\mathfrak{P}$, no point of intersection. For, if intersecting in a point $\mathfrak{z}$, they are produced from this point $\mathfrak{z}$ by the same quadratic cone, hence they are identical.

Therefore the quadratic cone K contains only one singular point, its centre $\mathfrak{P}$, which really consists of two singular points, infinitely near together, on the generatrix P.

The reciprocal surface of cone K is conic $\mathfrak{K}$.

Assume at random two cubics C_1 and C_2 of the 4-dimensional system,

$$\mathfrak{P}P\Pi.$$

They are produced from $\mathfrak{P}$ by two quadricones K_1 and K_2, which have a common generatrix P with common tangent plane Π, and contain each a quadratic pencil of self-reciprocal cubics.

These quadricones intersect in two farther rays, d and d'.

Through a point $\mathfrak{d}$ of d passes one curve of pencil K_1 and one curve of pencil K_2. These curves intersect d' in $\mathfrak{d}_1'$ and $\mathfrak{d}_2'$.

The osculating plane δ of point $\mathfrak{d}$ intersects P in point $\mathfrak{p}$. Then from $\mathfrak{p}$ both cubics C_1 and C_2 are projected by the same cubic cone with cuspidal-generatrix P.

§9.—*The Cubic Pencil of Self-reciprocal Cubics.*

The cubic pencil of self-reciprocal cubics is determined by three points, $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, as common points of intersection, or three planes, α, β, γ, as common planes of osculation.

Let $\pi = [\mathfrak{a}, \mathfrak{b}, \mathfrak{c}]$ and $\mathfrak{p} = (\alpha, \beta, \gamma)$. Then π is the nulplane of $\mathfrak{p}$ and therefore passes through $\mathfrak{p}$.

All the curves C of the cubic pencil are produced from $\mathfrak{p}$ by a cubic cone K with double generatrix d and the planes α, β, γ as inflexion-planes, and the osculating planes of all the cubics C envelop a curve $\mathfrak{C}$ of third class in π with a double tangent d', which, by the nulsystem, corresponds to the double generatrix d, while the whole curve $\mathfrak{C}$ is reciprocal to the cone K.

Proof: Any two twisted cubics which pass through $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ are produced from $\mathfrak{p}$ by two cubic cones with double generatrix, which have in common the three inflexion planes α, β, γ with their inflexion generatrices $|\mathfrak{p}\mathfrak{a}|$, $|\mathfrak{p}\mathfrak{b}|$, $|\mathfrak{p}\mathfrak{c}|$, and therefore are identical, q. e. d.

The cubic cone K has $\mathfrak{p} = (\alpha\beta\gamma)$ as centre, α, β, γ as inflexion planes with $a = |\mathfrak{p}\mathfrak{a}|$, $b = |\mathfrak{p}\mathfrak{b}|$, $c = |\mathfrak{p}\mathfrak{c}|$ as inflexion generatrices, and d as double generatrix. Hence it is of the 4th class.

The plane curve $\mathfrak{C}$ of 3rd class has the points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ as cuspidal points with a, b, c as cuspidal tangents; d', the conjugate ray of d, as double tangent, and therefore is of 4th order.

Every ray r which is drawn through $\mathfrak{p}$ in π contains 4 points of intersection with $\mathfrak{C}$, which are the nulpoles of the 4 tangent planes drawn through r onto K.

All the osculating planes of the cubics C of the pencil envelop $\mathfrak{C}$. Hence all the tangents of the curves C intersect $\mathfrak{C}$. All the tangents of C in the points $\mathfrak{x}$ of a generatrix x of cone K lie in a plane ξ which intersects $\mathfrak{C}$ in $\mathfrak{q}$. Hence all the tangents of C in the points $\mathfrak{x}$ of generatrix x pass through point $\mathfrak{q}$ and thereby constitute a plane pencil of rays.

The tangents of C in the two tangent planes δ_1 and δ_2 of the double generatrix d pass through the tangent points $\mathfrak{d}_1'$ and $\mathfrak{d}_2'$ of the double tangent d' of $\mathfrak{C}$.

Through every point $\mathfrak{d}$ of the double generatrix d pass two cubics C, and every plane δ' through the double tangent d' osculates two cubics C.

Through every point of the cone K passes one, and only one, cubic C, with the exception of the points of the double generatrix, through which pass two cubics C. Hence the cone K contains no singular points but $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, through which pass all the curves C of the pencil, with their singular planes α, β, γ, as common osculating planes of all the curves of the pencil.

The tangents of all the curves C constitute a system of rays of 4th order and class, with $\mathfrak{abc}$ as base points and $\alpha\beta\gamma$ as base planes.

From all the points $\mathfrak{x}$ of a generatrix x of the cone K the cubics C resp. are produced by quadricones upon the plane π in one and the same conic curve X.

Proof: Producing from two points $\mathfrak{x}_1$ and $\mathfrak{x}_2$ of x the two cubics C_1 and C_2, which pass through $\mathfrak{x}_1$ and $\mathfrak{x}_2$ resp. by quadricones, these quadricones intersect plane π in two conics X_1 and X_2, which have in common the three singular points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, the point $\mathfrak{x}'$ as point of intersection of the tangents of C in the points of x, or nulpol of ξ, and the tangent x' in $\mathfrak{x}'$ as conjugate ray to x. Hence these two conics X_1 and X_2 are identically the same.

Therefore all the ∞^2 quadricones passing through the twisted cubics C of the cubic pencil intersect π in a conic *pencil* of second order—that is, through every point of π pass 2 of these conics—and fourth class, that is, every straight line in π touches 4 of these conics (because from every point of π passes one chord through any C, but through every line of π four tangents on any C), and this pencil has as envelop the curve $\mathfrak{C}$ of 4th order and 3rd class, with one double tangent d' and three cuspidal points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$.

This conic pencil is projective to the rays of the cubic cone K, and to every generatrix of K corresponds a tangent plane ξ, a tangent point $\mathfrak{x}'$ on curve $\mathfrak{C}$, a tangent $t_{\mathfrak{x}}'$ of $\mathfrak{C}$ and a conic X touching $\mathfrak{C}$ in $\mathfrak{x}'$.

Assume two rays x and y of the cubic cone K and connect their corresponding points of intersection with the cubics C. These connecting lines $|\mathfrak{x}\mathfrak{y}|$ are common generatrices of those quadricones which produce C from the points of x, and those which produce C from the points of y. Hence they must intersect both projection conics X and Y in π, hence pass through one point. That is,

"*The connecting lines of the corresponding points of intersection of two generatrices of* K *with the cubics C form a linear pencil of rays with its centre in π,*" and

"*All the twisted cubics C of the cubic pencil intersect the rays of the cubic cone* K *in perspective ranges of points, which have their centres of perspectivity in π.*"

"*All the cubics C of the cubic pencil can be produced as the* (*partial*) *curves of intersection of two quadricones which have the points of intersection of a pencil of rays, with centre in π, with two lines x and y as centres, and produce two conics X and Y of the nulplane π of $\mathfrak{p} = (xy)$.*"

The cubic pencil is a configuration dual to itself, and so by inverting points and planes, we derive by the laws of duality the properties—

"*All the osculating planes of C which pass through a given tangent t' of $\mathfrak{C}$ are cut by the other osculating planes of C in conics, which from $\mathfrak{p}$ are projected by one and the same quadricone. This quadricone has the planes α, β, γ as tangent planes, and touches the cubic cone* K *in that generatrix t, which corresponds in the nulsystem to tangent t' of $\mathfrak{C}$.*"

"*From any two tangents t'_1 and t'_2* of $\mathfrak{C}$ *can be produced two osculating planes ξ_1 and ξ_2 onto each cubic C. These pairs of osculating planes ξ_1 and ξ_2 intersect each other in the rays of a plane pencil, which has the point of intersection of the tangents t'_1 and t'_2 as centre, and lies in the common tangent plane of the two quadricones with centres in $\mathfrak{p}$, which correspond to the tangents t'_1 and t'_2.*"

"On the double generatrix d of K exist two coincident projective ranges of points of intersection with the cubics C, which ranges of points have only one self-corresponding point $\mathfrak{p}$."

"The double tangent d' of $\mathfrak{C}$ is the centre of two coincident projective pencils of osculating planes of the cubics C, which pencil has only one self-corresponding plane π."

"Amongst the cubics C of the pencil exists one special cubic, which breaks up into three straight lines a, b, c."

"The cubics C are produced from the tangents t' of $\mathfrak{C}$ by perspective pencils of osculating planes," etc.

To every cubic C_1 of the cubic pencil can be found two other cubics C_2 and

C_3 of the pencil which intersect C_1, besides in the points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, still in a fourth point $\mathfrak{p}$ and $\mathfrak{p}'$ resp., on the double generatrix d. For C_1 intersects d in two points $\mathfrak{p}$ and $\mathfrak{p}'$, and through either one of these points passes another cubic C_2 and C_3 resp.

Let C_1 and C_2 be two cubics of the pencil $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ which have one further point $\mathfrak{d}$ in common. Let δ be the nulplane of $\mathfrak{d}$:

Then we can consider C_1 and C_2 as two cubics of the pencil

$$\begin{array}{llll}
(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) = |\alpha, \beta, \gamma|, & \text{with the centre } \mathfrak{p} = (\alpha, \beta, \gamma) & \text{and the double ray} & d = |\mathfrak{p}\mathfrak{d}|, \\
(\mathfrak{a}, \mathfrak{b}, \mathfrak{d}) = |\alpha, \beta, \delta|, & \text{`` `` ``} \quad \mathfrak{p}' = (\alpha, \beta, \delta) & \text{`` ``} & d' = |\mathfrak{p}'\mathfrak{c}|, \\
(\mathfrak{a}, \mathfrak{d}, \mathfrak{c}) = |\alpha, \delta, \gamma|, & \text{`` `` ``} \quad \mathfrak{p}'' = (\alpha, \delta, \gamma) & \text{`` ``} & d'' = |\mathfrak{p}'\mathfrak{b}|, \\
(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = |\delta, \beta, \gamma|, & \text{`` `` ``} \quad \mathfrak{p}''' = (\delta, \beta, \gamma) & \text{`` ``} & d''' = |\mathfrak{p}''''\mathfrak{a}|.
\end{array}$$

Hence the points $\mathfrak{abcd}$ are interchangeable, and 2 cubics, which have 4 common points of intersection, lie in 4 cubic pencils; that is, on 4 cubic cones. Hence two such cubics are *quadruply perspective.*

"*To every cubic C_1 of a cubic pencil can be found two other cubics C_2 and C_3 of the same cubic pencil, which are quadruply perspective to C_1.*"

"*Two cubics C_1 and C_2 of a nulsystem, which have* 4 *common points of intersection* $\mathfrak{abcd}$, *are quadruply perspective, and the* 4 *centres of perspectivity are the* 4 *vertices of that tetrahedron* $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$, *which, in the nulsystem, is conjugate to the tetrahedron* $\mathfrak{abcd}$. *The osculating planes of these two cubics can be put in correspondence with each other in four ways, so that corresponding osculating planes intersect each other in rays of a plane. These* 4 *planes, which give with the* 2 *cubics the same trace, are the sides of the tetrahedron of common points of intersection—*

$$\pi = [\mathfrak{abc}], \ \pi' = [\mathfrak{abd}], \ \pi'' = [\mathfrak{adc}], \ \pi''' = [\mathfrak{dbc}],$$

and pass through $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$ *resp.*"

"*To any cubic C exist in the nulsystem ∞^3 other cubics, which are quadruply perspective to C.*"

But these four centres of perspectivity of two quadruply perspective cubics can never be all real points.

For, as known, if the three osculating planes α, β, γ of a cubic C, which can be produced through a point $\mathfrak{p}$, are real, the chord d of C through $\mathfrak{p}$ intersects C in two imaginary points $\mathfrak{d}_\iota$ and $\mathfrak{d}'_\iota$, and if the chord d of a cubic C through a point $\mathfrak{p}$ intersects C in two real points $\mathfrak{d}$ and $\mathfrak{d}'$, two of the three osculating planes of C through $\mathfrak{p}$ are imaginary, β_ι, γ_ι.

Hence of the 4 points of intersection, $\mathfrak{abcd}$, and their osculating planes, $\alpha\beta\gamma\delta$, in the first case, $\mathfrak{d}_i$ and δ_i are imaginary;
in the latter case, $\mathfrak{b}_i\mathfrak{c}_i$ and $\beta_i\gamma_i$ are imaginary;
hence, of the points $\mathfrak{p}\mathfrak{p}'\mathfrak{p}''\mathfrak{p}'''$ and the planes $\pi\pi'\pi''\pi'''$
in the first case, $\mathfrak{p}_i'\mathfrak{p}_i''\mathfrak{p}_i'''$ and $\pi_i'\pi_i''\pi_i'''$ are imaginary;
in the latter case, $\mathfrak{p}_i'\mathfrak{p}_i''$ and $\pi_i'\pi_i''$ are imaginary.

That means, of the 4 centres of perspectivity of 2 quadruply perspective cubics either two are imaginary and two real, the cubics contain two real and two imaginary points of intersection, give the same trace on two real and two imaginary planes, and have two real and two imaginary common osculating planes, or

Three centres of perspectivity are imaginary and only one real, the cubics contain three real and one imaginary point of intersection, give the same trace on one real and three imaginary planes, and have three real and one imaginary osculating planes in common. Hence,

The highest number of real points of intersection or of common osculating planes of cubics of a nulsystem is two, and two cubics of a nulsystem can never be more than double perspective with two real centres of perspectivity.

Herefrom we derive the result—

"*If two cubics of the same linear nulsystem have* 4 *points*, $\mathfrak{abcd}$ *in common, they have* 4 *common osculating planes also*, $\alpha\beta\gamma\delta$, *and viz. the tetrahedron of the four common points of intersection*, T, *and the tetrahedron of the four common osculating planes*, T', *are conjugate tetrahedra of the nulsystem. The* 4 *planes of the tetrahedron of the common points of intersection* T *are cut by the osculating planes of the two cubics in the tangents of the same quartic curve of* 3rd *class, and the points of both cubics are projected from the* 4 *vertices of the tetrahedron of common osculating planes*, T', *by the same cubic cones of* 4th *class.*

"*Of the common points of intersection, and of the common osculating planes, not more than three and not less than two are real; of the tracing curves in the planes of the tetrahedron* T, *and of the projecting cones from the vertices of tetrahedra* T', *or centres of perspectivity of both cubics, not more than two and not less than one are real—the others imaginary.*"

The general cubic pencil contains 0, 2 or 4 cubic hyperbolic parabolas, according to the number of real points of intersection of $\mathfrak{C}$ with the line of infinity, but no cubic parabola, except if the cubic cone K passes through the pole $\mathfrak{p}_\infty$ of the plane at infinity, π_∞. In this case $\mathfrak{C}$ is a parabolic curve; that is, has the line of infinity as tangent, and on the ray $|\mathfrak{p}\mathfrak{p}_\infty|$ the tangents of all the cubics C are parallel.

§10.—*Special Self-reciprocal Cubics and their Configurations.*

I.—PARTICULAR POSITIONS AND COINCIDENCE OF BASE POINTS.

1). If two of three base points $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of the cubic pencil lie upon a self-conjugate ray, $s = |\mathfrak{b}\mathfrak{c}|$, their osculating planes β and γ pass through the same ray, $s = |\beta\gamma|$, the centre of the cubic cone K is the point of intersection of s with α, the osculating plane of $\mathfrak{a}$, $\mathfrak{p} = (s\alpha)$, and the plane of the curve $\mathfrak{C}$ is $\pi = |s\mathfrak{a}|$. The cubics C, because of having 4 points of intersection with the planes β and γ, the triple point $\mathfrak{b}$ and $\mathfrak{c}$ with β, and the triple point $\mathfrak{c}$ and $\mathfrak{b}$ with γ, must break up into the double line s and the pencil of self-conjugate rays in α. The cubic cone K consists of the three planes $\alpha\beta\gamma$ and the curve $\mathfrak{C}$ of the four lines, $|\mathfrak{a}\mathfrak{b}|$, $|\mathfrak{b}\mathfrak{c}|$, $|\mathfrak{c}\mathfrak{a}|$, $|\mathfrak{a}\mathfrak{p}|$.

2). If two of the three base points $\mathfrak{b}$ and $\mathfrak{c}$ coincide into one point, the connecting line of these two coincident points, $s = |\mathfrak{b}\mathfrak{c}|$, is either self-conjugate ray or not.

If $s = |\mathfrak{b}\mathfrak{c}|$ is no self-conjugate ray, the cubic pencil can contain no real cubics, because every cubic of the pencil must pass through $\mathfrak{b} = \mathfrak{c}$ without having $s = |\mathfrak{b}\mathfrak{c}|$ as tangent. Hence, if s' is the conjugate ray of s, $\mathfrak{p}$ is the point of intersection of s' with α, $\mathfrak{p} = (s'\alpha)$, and lies upon $\pi = [\mathfrak{a}s]$. The cubic cone K consists of the double plane $[s'\mathfrak{b}]$ and the plane α; the curve $\mathfrak{C}$ of $|\mathfrak{a}\mathfrak{p}|$, $s = |\mathfrak{b}\mathfrak{c}|$ and the double line $|\mathfrak{p}\mathfrak{b}| = |\mathfrak{p}\mathfrak{c}|$. The cubics C break up into the self-conjugate rays of pencil $(\mathfrak{a}\alpha)$ and the double rays of pencil $|\mathfrak{b}s'| = |\mathfrak{c}s'|$.

If $s = |\mathfrak{b}\mathfrak{c}|$ is a self-conjugate ray of the nulsystem, the pencil consists of those twisted cubics which pass through $\mathfrak{a}$ and $\mathfrak{s} = \mathfrak{b} = \mathfrak{c}$, and have in $\mathfrak{s}$ the common tangent s. The centre of the cone K is the point of intersection $\mathfrak{p} = (s\alpha)$, is of the 3rd order and 3rd class, with s as cuspidal generatrix and α as inflexion plane. The curve $\mathfrak{C}$ lies in the plane $\pi = [\mathfrak{a}s]$, is of 3rd order and 3rd class, with s as inflexion tangent and $\mathfrak{a}$ as cuspidal point.

3). If all the base points $\mathfrak{a}\mathfrak{b}\mathfrak{c}$ coincide into one point, this point $\mathfrak{p}$ is the centre of the projection cone K, all the cubics C osculate each other in $\mathfrak{p}$, and the cone K is a quadricone while the curve $\mathfrak{C}$ in π is a conic curve. The cubics

C of this pencil have, besides the triple point $\mathfrak{p}$, no further points in common, and the pencil is the quadratic pencil treated in §8, which in this way appears as a special case of the general cubic pencil.

II.—Cylinder Pencils.

4). If the centre $\mathfrak{p}$ of the cone K lies in the plane at infinity, the cone K becomes a cylinder of 3rd order and 4th class. This is the case if plane $\pi = |\mathfrak{abc}|$ passes through the pole $\mathfrak{P}_\infty$ of the plane at infinity.

Such cylinder pencils exist ∞^8.

5). The most interesting of them are those which have the pole of the plane at infinity, $\mathfrak{P}_\infty$ as centre.

Their curve $\mathfrak{C}$ lies entirely in the infinity. Hence all the pencils of tangents and of connecting lines of the points of intersection of the cubics C with two generatrices x and y are parallel systems of lines. Therefore,

"All the cubics of a pencil with $\mathfrak{P}_\infty$ as centre are congruent and parallel."

"To any cubic of the nulsystem exists an infinite number of congruent and parallel cubics which lie upon a cylinder with $\mathfrak{P}_\infty$ as centre."

For every cubic is produced from $\mathfrak{P}_\infty$ by a cubic cylinder.

Such pencils exist ∞^6.

Their singular points $\mathfrak{abc}$ lie in the plane at infinity.

Hence

"*Every self-reciprocal curve of the nulsystem can be transferred parallel with itself in the direction to and fro the pole of the plane at infinity without ceasing to be self-reciprocal.*"

6). Amongst the cylinder pencils with $\mathfrak{P}_\infty$ as centre exist ∞^4 parabolic pencils, which consist of an infinite number of congruent and parallel cubic parabolas on a quadratic parabolic cylinder.

Hence the nulsystem contains ∞^5 self-reciprocal cubic parabolas.

7). Every point at infinity determines a quadratic cylindric pencil.

§11.—*The Quartic Pencil.*

I.—On the Cubic Cone.

Produce a cubic cone K with double generatrix, with a point $\mathfrak{p}$ as centre, which has the nulplane π of $\mathfrak{p}$ as tangent plane, with p as tangent generatrix.

Then this plane π intersects cone K in one further generatrix a. Let this generatrix a be inflexion-generatrix of K, with plane α as its inflexion-plane, and point $\mathfrak{a}$ as inflexion-point, a is self-conjugate ray, and α passes through $\mathfrak{p}$.

Then points $\mathfrak{p}$ and $\mathfrak{a}$ are singular points of the cone K, and therefore common points of intersection of all its self-reciprocal curves.

The two other inflexion-generatrices b and c lie in a plane ϕ with a, and the nulplanes of b and c, β and γ resp., form two plane pencils with b' and c' as axes. b' and c' lie in π, but none of the planes β and γ passes through $\mathfrak{p}$; hence they intersect the inflexion-planes β_1 and γ_1 of cone K length b and c in three points of a straight line, infinitely near with each other. That means, the tangents of the self-reciprocal curves on K, length b and c, are stationary tangents.

Let d be the double generatrix of K, with δ_1 and δ_2 as tangent planes.

The conjugate rays x' of the generatrices x of K lie in the plane π and envelop there a curve $\mathfrak{C}$ of 3rd class and 4th order, which passes through $\mathfrak{p}$ and $\mathfrak{a}$, and has in $\mathfrak{p}$ the tangent p. This curve has the points $\mathfrak{a}\mathfrak{b}'\mathfrak{c}'$ as cuspidal point, the rays $ab'c'$ as cuspidal tangents, and d', the conjugate ray of d, as double tangent with $\mathfrak{d}_1$ and $\mathfrak{d}_2$ as tangent points.

Through every point $\mathfrak{x}$ of K passes one, and only one, self-reciprocal curve which has the connecting line x_1 of $\mathfrak{x}$ with the point of intersection $\mathfrak{x}'$ of ξ and $\mathfrak{C}$, the nulpol of ξ, as tangent. In the infinitely near tangent plane ξ_1 the point of the self-reciprocal curve is derived again by its tangent x_2 intersecting $\mathfrak{x}_1$ and $\mathfrak{C}$, etc., and so the self-reciprocal curve C is produced by a line which, lying in a tangent plane and intersecting curve $\mathfrak{C}$ in the nulpol of this tangent plane, describes a developable surface.

This curve C intersects plane π in $\mathfrak{a}$ and the triple point $\mathfrak{p}$, where π osculates C. Any plane ε through $\mathfrak{p}$ intersects K in three rays x, y, z, and each of these contains a point $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ of C, while $\mathfrak{p}$ is the fourth point of intersection of ε with C.

Hence C is of the 4th order.

From any point $\mathfrak{q}$ of π pass three tangents $x'y'z'$ onto $\mathfrak{C}$, which each determine an osculating plane of C. π is osculating plane also; hence

C is of 4th class.

The osculating developable of C intersects the osculating plane π in the curve $\mathfrak{C}$ of 4th order, and the tangent p, counting twice. Hence

C is of 6th rank.

A stationary tangent must be produced from $\mathfrak{p}$ by a stationary or inflexion plane. K contains two such planes (besides α), β and γ, Hence

C has two stationary tangents, etc.

Therefore,

"*C is the rational quartic with two stationary tangents,*" and

"*Cone* K *contains a pencil of self-reciprocal quartics which pass through* $\mathfrak{a}$ *and osculate each other in* $\mathfrak{p}$."

"*d is the one trisecant chord of C.*"

"*From all the points* $\mathfrak{r}$ *of any generatrix* ξ_1 *of* K *all the quartics are produced upon* π *by one and the same plane cubic with double points.*

Hence, given one twisted quartic C_1 on K, all the others can be produced by producing the curve of intersection of π with that cubic cone which projects C_1 from one of its points $\mathfrak{r}_1$, from all the other points of the same generatrix ξ_1."

Besides in point $\mathfrak{p}$, the double ray d intersects any self-reciprocal quartic C in two points $\mathfrak{d}_1$ and δ_2. Producing C from $\mathfrak{d}_1$, we derive a cone Δ_1 which has d as double generatrix, and in d, plane δ_2 as common tangent plane with K, while it has plane $[\mathfrak{d}_2 p]$ as other tangent plane in d. Hence, its other curve of intersection is only C.

Through any point $\mathfrak{d}$ of d pass two quartics C_1 and C_2. The one has its tangent in δ_1, the other in δ_2. They are produced from $\mathfrak{d}$ by cubic cones which have d as double generatrix with one common tangent plane $[\mathfrak{d}p]$ and as other tangent-plane δ_2 and δ_1 resp.

Remark: A rational cubic cone contains a *quartic* pencil if the nulplane of its centre is tangent-plane, and, besides this, intersects in an inflexion generatrix; a rational cubic cone contains a *cubic* pencil if the nulplane of its centre intersects in three inflexion generatrices.

II.—QUARTIC PENCIL ON A QUARTIC CONE.

Projecting a self-reciprocal quartic C from an outside point $\mathfrak{p}$ by a rational quartic cone K, we derive on K a pencil of self-reciprocal quartics.

Perspective-cone K contains three double rays $d_1 d_2 d_3$. The nulplane π of $\mathfrak{p}$ intersects C in four singular points $\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}$, with their nulplanes $\alpha\beta\gamma\delta$ as inflexion planes, and $abcd$ as inflexion generatrices. K contains two farther inflexion planes $\tau_1\tau_2$ with inflexion generatrices $t_1 t_2$, which contain the stationary tangents of the quartics. The cone K is of 4th order and of 6th class, and the tangents of all the self-reciprocal quartics intersect π in the points of a curve $\mathfrak{C}$ of 6th order and 4th class, with six cuspidal points $\mathfrak{a}\mathfrak{b}\mathfrak{c}\mathfrak{d}\, t_1 t_2$ and three double tangents $d_1' d_2' d_3'$.

This pencil of quartics shares the property, that all the generatrices of its perspective cone K are intersected by the quartics in perspective ranges of points, with their centre of perspectivity in π; all the tangents of C along a generatrix x form a plane pencil with centre on $\mathfrak{C}$; all the quartics C are produced from the points $\mathfrak{r}$ of a generatrix x by cubic cones which intersect π in one and the same cubic curve of 4th class, etc.

The greatest number of common points of intersection of two self-reciprocal quartics or their greatest number of common osculating planes is five, four of which must lie in a plane π, the nulplane of the centre of their perspective cone of 4th order K, which sends a double generatrix through the fifth common point.

§12.—*The n^{tic} Pencil.*

The self-reciprocal n^{tic} can be construed as partial curve of intersection of two rational cones of $(n-1)^{\text{th}}$ order, which have their centres on the n^{tic} C, or of two rational cones of n^{th} order.

For, the constants of the n^{tic} being given in §2, the constants of its projecting cone of $(n-1)^{\text{th}}$ order from one of its points are known, and on this cone of $(n-1)^{\text{th}}$ order K an infinite number of n^{tics} can be produced either by the infinitesimal method or by intersection with $(n-1)^{\text{tic}}$ cones with the centre in one of the generatrices of K.

K contains a pencil of self-reciprocal n^{tics}.

From an outside point $\mathfrak{p}$ a self-reciprocal n^{tic} C is produced by a rational perspective-cone K of n^{th} order and $2(n-1)^{\text{th}}$ class, with $\dfrac{(n-1)(n-2)}{2}$ double-generatrices, and $n + 2(n-3) = 3(n-2)$ inflexion planes K generatrices, n of which lie in the nulplane π of the centre $\mathfrak{p}$, the other $2(n-3)$ produce the stationary tangents of C.

This cone K contains a pencil of self-reciprocal n^{tics} C which intersect each other in n singular points $\mathfrak{a}, \mathfrak{b} \ldots.$ of the plane π, the nulpoles or osculating points of the singular planes or common osculating planes $\alpha, \beta \ldots.$ which intersect each other in $\mathfrak{p}$.

Two self-reciprocal n^{tics} never contain more than $(n+1)$ common points of intersection, with common osculating planes. n of these points always lie in a plane π, while n of the common osculating planes pass through a point $\mathfrak{p}$.

All the self-reciprocal n^{tics} C of the pencil intersect the generatrices of cone K in perspective ranges of points, with their centres of perspectivity in plane π, the nulplane of centre $\mathfrak{p}$.

The tangents of all the n^{tics} of the pencil intersect plane π in the points of a curve $\mathfrak{C}$ of $2(n-1)^{\text{th}}$ order in π and n^{th} class in π, with $3(n-2)$ cuspidal points, $\mathfrak{a}$, $\mathfrak{b}$, ... and the traces of the $2(n-3)$ stationary tangents, and with $\frac{(n-1)(n-2)}{2}$ double tangents.

All the tangents of C along a generatrix x form a plane pencil with centre on $\mathfrak{C}$.

All the n^{tics} C are projected from their points of intersection $\mathfrak{x}$ with a generatrix x by cones of $(n-1)^{\text{th}}$ order and $2(n-2)^{\text{th}}$ class, which intersect π in one and the same curve of $(n-1)^{\text{th}}$ order.

All the n^{tics} C are projected from all their points upon the plane π by the curves of $(n-1)^{\text{th}}$ order of a 1-dimensional system of $(n-1)(n-2)^{\text{th}}$ order and $2(n-1)(n-2)^{\text{th}}$ class, with the curve of $2(n-2)^{\text{th}}$ order and $(n-1)^{\text{th}}$ class $\mathfrak{C}$ as envelope.

Hence all the curves of the n^{tic} pencil can be produced as the curves of intersection of $2(n-1)^{\text{tic}}$ cones, Ξ and H, which have their centres in the points $\mathfrak{x}$ and $\mathfrak{y}$ resp. of two generatrices x and y of K, have the rays of the pencil in plane $[xy]$ with centre in π resp. in common, and project the same two curves of $(n-1)^{\text{th}}$ order and $2(n-2)^{\text{th}}$ class in π resp.

Multiple perspectivity does not exist amongst higher curves than cubics.

CONTENTS.

§1. Introduction.. 161
§2. Self-reciprocal Algebraic Curves in the Nulsystem........................ 163
§3. Self-reciprocal Straight Lines and their Configurations in the Nulsystem.... 166
§4. Self-reciprocal Cubic Curves in the Nulsystem............................ 167
§5. Self-reciprocal Curves on given Surfaces................................. 168
§6. Self-reciprocal Curves on Ruled and Developable Surfaces................. 168
§7. Self-reciprocal Curves on Ruled Hyperboloids............................. 171
§8. The Quadratic Pencil of Self-reciprocal Cubics........................... 173
§9. The Cubic Pencil of Self-reciprocal Cubics............................... 175
§10. Special Self-reciprocal Cubics and their Configurations................. 180
    I. Particular Position and Coincidence of Base-points.
    II. Cylinder Pencils.
§11. The Quartic Pencil.. 181
    I. On a Cubic Cone.
    II. On a Quartic Cone.
§12. The n^{tic} Pencil.. 184

A Classification of Logarithmic Systems.[*]

By Irving Stringham.

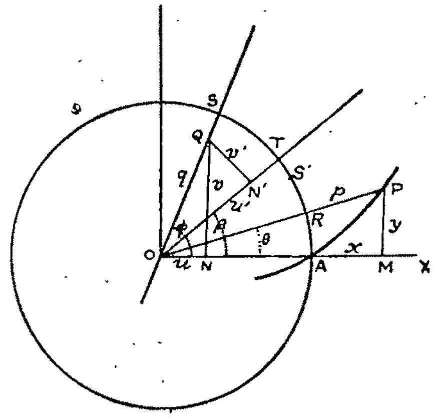
Though the graphical representation of logarithms by means of the logarithmic spiral is well known, I am not aware that any attempt has been made to use this curve, properly defined as a geometrical locus, as the means for defining the logarithm and demonstrating its properties. The problem turns out to have not only its geometrical interest, but also some importance for analysis in general, giving rise to what I venture to call *gonic* systems of logarithms, whose moduli contain an angular determining-element, and leading, through their introduction, to a classification of logarithmic systems.

Construction and Definitions: The Gonic Systems.—In a circle whose radius is unity, the diameters through S and T are fixed, OR turns about O with a constant velocity, Q moves along OS with a constant velocity, P along OR with a velocity proportional to its distance from O and is supposed to cross the circumference of the circle at A at the instant when Q passes O. Let the velocities of

$$\begin{aligned} &P \text{ in } OR \text{ at } A = \lambda, \\ &Q \text{ in } OS \quad = \mu, \\ &R \text{ in } ARS \quad = \omega. \end{aligned}$$

Let

$$\begin{array}{lll} OM = x, & MP = y, & OP = p, \\ ON = u, & NQ = v, & OQ = q, \\ ON' = u', & N'Q = v', & \text{arc } AS = \phi, \\ \mu/\sqrt{\lambda^2+\mu^2} & = m, & \text{arc } AT = \beta, \\ \multicolumn{3}{l}{\overline{OT} = \cos\beta + i\sin\beta = \text{cis}\,\beta = \varepsilon.} \end{array}$$



OX is taken as the real axis and

$$z = x + iy, \quad w = u + iv, \quad i = \sqrt{-1}.$$

The condition $\tan(\phi - \beta) = \omega/\lambda$ is assigned arbitrarily; and this, together with $\mu/\sqrt{\lambda^2 + \omega^2} = m$, by elimination of ω, gives also

$$m = \mu/\lambda \,.\, \cos(\phi - \beta).$$

---

[*] A paper read before the New York Mathematical Society, November 3d, 1891.

Thus m—as is seen from its value in terms of λ, μ and ω—is the ratio of the velocity of Q in OS to that of P in AP at A, and—as its value $\mu/\lambda . \cos(\phi - \beta)$ shows—it is also the ratio of the velocity of N' along OT to that of P along OR when R is at A.

Since the rates of variation of u' and p, so long as m remains constant, maintain their ratio to each other undisturbed, namely, the ratio m/p, the fixing of m fixes their relative values. Hence m, and β, which is independent of m, are assumed to be constant for all corresponding values of z, w, in any one system of logarithms, whatever the value of ϕ, and fix the system, while λ, μ, ω may vary subject only to the conditions

$$\mu/\sqrt{\lambda^2 + \omega^2} = \text{constant}, \ \tan(\phi - \beta) = \omega/\lambda.$$

The terms modulus, logarithm, base, and exponential are defined as follows:

1. $m\varepsilon$ is the modulus of the system of logarithms.

2. The difference $\overline{OQ} - \overline{OQ'}$ is the logarithm, with respect to modulus $m\varepsilon$, of the ratio $\overline{OP}/\overline{OP'}$, wherein P, P' correspond to Q, Q'. In particular $\overline{OQ}$ is the logarithm, to modulus $m\varepsilon$, of $\overline{OP}$. The symbolic definitions are:

$$w - w' \equiv \log^{(m\epsilon)}(z/z');$$
$$w \equiv \log^{(m\epsilon)} z.\text{*}$$

By assuming that P passes A at the instant when Q passes the origin, so that to $u = v = 0$ corresponds $y = 0$, $x = 1$, we take advantage of the convenient relation

$$\log^{(m\epsilon)} 1 = 0.$$

3. In any system of logarithms the value of z, for which $w = 1$, is the base of the system; it is here denoted by b_ε.

4. The value of p, for which $u' = 1$, will remain unchanged and independent of β so long merely as m is constant.

Denoting this value of p by b, we define z symbolically as a function of w by the identity

$$z \equiv b_\epsilon^w.$$

The expression b_ϵ^w is here called the exponential of w with respect to base b_ϵ.

---

[*] I have elsewhere written this in the form ${}^{m\epsilon}\log z$, but this notation is sometimes used to denote logarithm with respect to *base* $m\epsilon$ (see Harnack, Diff. u. Int. Rechnung, p. 18). It seems best therefore to abandon it in favor of a notation that has not been already used by reputable authors for another purpose.

The Law of Metathesis.—The value of m being unchanged, suppose μ changed to $k\mu$ (k a real quantity), then $\mu/\sqrt{\lambda^2+\omega^2}.\varepsilon$ becomes $k\mu/\sqrt{\lambda^2+\omega^2}.\varepsilon = km\varepsilon$, and q, w become kq, kw, whence

$$\begin{aligned} kw &= k\log^{(m\varepsilon)}z \\ &= \log^{(km\varepsilon)}z, \\ \text{and } \therefore \text{ also } \quad &= km\log^{(\varepsilon)}z. \end{aligned}$$

More particularly

$$w = \log^{(m\varepsilon)}z = m\log^{(\varepsilon)}z,$$

from which, by putting $w = 1$, and correspondingly $z = b_\varepsilon$, we obtain m in the form

$$m = 1/\log^{(\varepsilon)}b_\varepsilon,$$

and the logarithmic function in the consequent form

$$w = \log^{(m\varepsilon)}z = \frac{\log^{(\varepsilon)}z}{\log^{(\varepsilon)}b_\varepsilon}.$$

When now for z in this last equation we substitute its exponential equivalent b_ε^w, the formula thus obtained,

$$w\log^{(\varepsilon)}b_\varepsilon = \log^{(\varepsilon)}b_\varepsilon^w,$$

expresses the law of metathesis, if I may so call it, for the natural system of logarithms with complex modulus. For logarithms in general the similar formula

$$w\log^{(k\varepsilon)}b_\varepsilon = \log^{(k\varepsilon)}b_\varepsilon^w.$$

is obtained from the preceding by multiplying each member by the real quantity k, the complex quantity b_ε representing, in the new system with the new modulus $k\varepsilon$, any complex value whatever.

The Law of Involution.—Successive applications of the law of metathesis give the following equivalent expressions:

$$\begin{aligned} w\log^{(k\varepsilon)}b_\varepsilon^{w'} &= w'\log^{(k\varepsilon)}b_\varepsilon^w \\ = \log^{(k\varepsilon)}(b_\varepsilon^{w'})^w &= \log^{(k\varepsilon)}(b_\varepsilon^w)^{w'}. \end{aligned}$$

A sufficient condition for the equivalence of the last two is

$$(b_\varepsilon^{w'})^w = (b_\varepsilon^w)^{w'},$$

and for every pair of values w, w', there exists the corresponding equation as above written, expressing the law of involution. We may therefore omit the parentheses and write without ambiguity

$$(b_\varepsilon^{w'})^w = b_\varepsilon^{w'w} = b_\varepsilon^{ww'}.$$

The Addition Theorem.—The definition of the logarithmic function states also in substance the addition theorem; for the equation of definition is

$$\begin{aligned} w - w' &= \log^{(m\epsilon)}z - \log^{(m\epsilon)}z' \\ &= \log^{(m\epsilon)}(z/z'), \end{aligned}$$

which is the form of the theorem for $w - w'$. In order to obtain the form for $w + w'$, we write in the last equation $w = 0$, whence

$$-w' = \log^{(m\epsilon)}(1/z');$$

but

$$\begin{aligned} +w' &= \log^{(m\epsilon)}z', \\ \therefore\ w + w' &= \log^{(m\epsilon)}z + \log^{(m\epsilon)}z' \\ &= \log^{(m\epsilon)}(z.z'), \end{aligned}$$

which is the form required.

The Index Law follows as a corollary from the addition theorem. Thus if

$$z = b_\epsilon^w \qquad , \quad z' = b_\epsilon^{w'},$$

$$\therefore\ w = \log^{(m\epsilon)}z,\ w' = \log^{(m\epsilon)}z'$$

and

$$z.z' = b_\epsilon^w . b_\epsilon^{w'}.$$

But

$$\begin{aligned} \log^{(m\epsilon)}(z.z') &= \log^{(m\epsilon)}z + \log^{(m\epsilon)}z' \\ &= w + w', \\ \therefore\ z.z' &= b_\epsilon^{w+w'} \\ &= b_\epsilon^w . b_\epsilon^{w'}, \end{aligned}$$

which is the index law for multiplication; and the law for division is deduced by a repetition of this process with the signs / and —.

The Agonic System: $\beta = 0$.—The special value zero for the modular angle AOT eliminates the imaginary term from the modulus and introduces the ordinary system of logarithms, whose equations of definitions are

$$w' = \log^{(m1)}z, \quad z = b_1^{w'},$$

or as they may now be written without ambiguity,

$$w' = \log^{(m)}z, \quad z = b^{w'}.$$

The original systems above described, whose modulus involves an independent angular element, will be here referred to as *gonic* systems; when specialized by the omission of this angular element it will be called the ordinary or *a-gonic* system.

The laws of operation in the a-gonic system have their expression in the formulae already deduced for the gonic systems, with β and ε everywhere dropped out. The geometrical representation is obtained by turning the figure $TN'OQS$ backward through the angle AOT, so that T, S fall into the positions A, S'. The new ϕ, say ϕ', $=$ arc $TS =$ arc AS', the new modulus

$$= \mu / \sqrt{\lambda^2 + \omega^2} = \mu / \lambda . \cos \phi' = m,$$

its former value with the factor ε omitted, no change in λ, μ, and ω having taken place. Thus the motion of P remains intact and the new Q moves in the line OS' with its former velocity. Hence the values of z in the two systems are identical, while w' in the agonic system and w in the gonic bear the relation to each other,

$$w = w' \operatorname{cis} \beta = w' \varepsilon,$$

and we may write

$$w = \log^{(m\epsilon)} z = \varepsilon \log^{(m)} z.$$

Since ε is independent of m, this equation, together with

$$k \log^{(m\epsilon)} z = \log^{(km\epsilon)} z$$

previously demonstrated, expresses the law: *To multiply the modulus of a logarithm by any quantity has the effect of multiplying the logarithm itself by the same quantity.*

The corresponding equation in terms of the exponential functions, namely,

$$b_\epsilon^w = b^{w/\epsilon},$$

gives us (by the law of involution) the value of the gonic base b_ϵ in the form

$$b_\epsilon = b^{1/\epsilon}.$$

The value of b itself, expressed in terms of m and e, the base corresponding to modulus 1, is obtained in like manner from the inverse of the equation $1/m = \log^{(\epsilon)} b_\epsilon$, namely, from

$$b_\epsilon = e_\epsilon^{1/m} = e^{1/m\epsilon},$$

giving

$$b = e^{1/m}.$$

A second agonic system is obtained by putting $\beta = \pi$. Its modulus is $-m$, its base $1/b$, and its logarithms are the negatives of those in the systems whose modulus and base are $+m$ and b. In every other respect the two systems are alike.

Linear Systems.—When the angle $TOS = 0$, the corresponding values of function and variable are $w = q \operatorname{cis} \beta = q\varepsilon$, and $z = p$ a real quantity; and the relation between p and q is expressed in either of the forms

$$w = q\varepsilon = \log^{(m\epsilon)} p\,, \quad z = p = b_{\epsilon}^{q\epsilon},$$

belonging, when β is not zero, to a gonic system. Q moves in OT with the velocity μ, P in OA with the velocity λp. Hence in a gonic system the logarithm of a real quantity is, in general, complex; though in particular, when $\beta = \pm \pi/2$, it is a pure imaginary,

When $\beta = 0$, the condition $\phi = \beta$ makes angle $AOT =$ angle $TOS = 0$, $w = q$, $z = p$, $\omega = 0$, $m = \mu/\lambda$; $\varepsilon = 1$; and the relations between p and q are

$$q = \log^{(m)} p\,, \quad p = b^q.$$

These may be regarded as belonging to the general agonic system, obtained from it by making $\phi = 0$, or as the defining equations of the ordinary system for real quantities only, whose geometrical representatives are lengths upon the real axis laid off in accordance with the logarithmic law: that q varies uniformly while the rate of change of p bears a constant ratio to its own length. The modulus, μ/λ, is the ratio of the rate of q to that of p at the instant when $p = 1$, and the base, b, is the value of p when $q = 1$. This is the well-known Naperian representation of the logarithmic function.

The Semi-cosine Equivalent of b_{ϵ}^{w}.—Returning to the general case: ϕ and β unequal and not zero, and denoting the arc AR by θ, $\overline{OP}$ is

$$z = p \operatorname{cis} \theta = b_{\epsilon}^{w}.$$

Then since the velocity of N' in OT is $\mu \cos(\phi - \beta)$ and that of P in OR is λp, while $m = \mu/\lambda . \cos(\phi - \beta)$, the relation between ON', $= u'$ and OP, $= p$, two real quantities, is that of an exponential to its logarithm, with respect to the modulus m; that is,

$$u' = \log^{(m)} p \text{ and } p = b^{u'}.$$

And again, since $\omega = \lambda \tan(\phi - \beta)$ and $m = \mu/\lambda . \cos(\phi - \beta)$,

$$\therefore\ m\omega = \mu \sin(\phi - \beta);$$

but ω and μ are the rates of change of θ and q respectively, and $q = 0$, $\theta = 0$ are simultaneous values;

$$\therefore\ m\theta = q \sin(\phi - \beta) = v'$$

and
$$\theta = \frac{v'}{m}.$$

Hence
$$b_\epsilon^w = p \operatorname{cis} \theta$$
$$= b^{u'} \operatorname{cis} \frac{v'}{m};$$

or also, introducing $m\varepsilon$ and b_ϵ in place of m and b, and substituting for w its equivalent $\varepsilon u' + i\varepsilon v'$,

$$b_\epsilon^{\varepsilon u' + i\varepsilon v'} = b_\epsilon^{\varepsilon u'} \operatorname{cis} \frac{\varepsilon v'}{m\varepsilon};$$

or again, replacing u', v' by their values, $u \cos \beta + v \sin \beta$, $v \cos \beta - u \sin \beta$, in terms of u, v, β,

$$b_\epsilon^{u + iv} = b^{u \cos \beta + v \sin \beta} \operatorname{cis} \frac{v \cos \beta - u \sin \beta}{m}.$$

When $\beta = 0$, this assumes the familiar form

$$b^{u + iv} = b^u \left(\cos \frac{v}{m} + i \sin \frac{v}{m} \right).$$

The Derivatives of the Logarithmic and Exponential Functions.—By definition, the rates of change of p, q and θ are respectively

$$\frac{dp}{dt} = \lambda p, \quad \frac{dq}{dt} = \mu, \quad \frac{d\theta}{dt} = \omega,$$

in which t is taken to represent time. Differentiating z and w, expressed in terms of p, q, θ and ϕ, we have

$$dz = d(p \operatorname{cis} \theta),$$
$$= (dp + ip d\theta) \operatorname{cis} \theta$$
$$= (\lambda p + ip\omega) \operatorname{cis} \theta . dt,$$

and
$$dw = d(q \operatorname{cis} \phi)$$
$$= dq . \operatorname{cis} \phi$$
$$= \mu \operatorname{cis} \phi . dt,$$
$$\therefore \frac{dw}{dz} = \frac{\mu \operatorname{cis} \phi}{p(\lambda + i\omega) \operatorname{cis} \theta},$$

and since $\omega = \lambda \tan (\phi - \beta)$,

$$\therefore \frac{dw}{dz} = \frac{\mu \operatorname{cis} \phi \cos (\phi - \beta)}{p\lambda \operatorname{cis} (\phi - \beta) \operatorname{cis} \theta}$$
$$= \frac{\mu \cos (\phi - \beta) \operatorname{cis} \beta}{\lambda z}$$
$$= \frac{m\varepsilon}{z},$$

the formula for the derivative of the logarithmic function. Hence also, for the exponential function,

$$\frac{dz}{dw} = \frac{z}{m\varepsilon}$$
$$= b_\varepsilon^w . \log^{(1)} b_\varepsilon ;$$

for it has been shown that $1/m = \log^{(\varepsilon)} b_\varepsilon = \varepsilon \log^{(1)} b_\varepsilon$.

The Locus of P.—The rates of change of p and θ are respectively λp and ω, $= \lambda \tan(\phi - \beta)$; hence by the definition of a logarithm, in real quantities, θ is the logarithm of p with respect to the modulus $\{\lambda \tan(\phi - \beta)\}/\lambda = \tan(\phi - \beta)$; or, in equivalent terms,

$$\theta = \tan(\phi - \beta) . \log^{(1)} p ,$$

which is the equation of the logarithmic spiral. Because $p d\theta / dp = \tan(\phi - \beta)$, this curve crosses its radius vector at an angle $= TOS$.

The Sine and Cosine Functions.—An obvious generalization of the formulae of the circular and hyperbolic functions is obtained by assuming, as definitions of the generalized sine and cosine, the functions

$$\frac{\varepsilon}{2}(b_\varepsilon^w - b_\varepsilon^{-w}), \quad \frac{1}{2}(b_\varepsilon^w + b_\varepsilon^{-w}).$$

Let these be denoted for the moment by $\text{sinb}_\varepsilon w$ and $\text{cosb}_\varepsilon w$ respectively. Well known substitutions and reductions then lead to the following formulae:

$$b_\varepsilon^w = \text{cosb}_\varepsilon w + \varepsilon^{-1} \text{sinb}_\varepsilon w ,$$
$$b_\varepsilon^{-w} = \text{cosb}_\varepsilon w - \varepsilon^{-1} \text{sinb}_\varepsilon w ,$$
$$\text{cosb}_\varepsilon^2 w - \varepsilon^{-2} \text{sinb}_\varepsilon^2 w = 1 ,$$
$$\text{sinb}_\varepsilon (w \pm w') = \text{sinb}_\varepsilon w \, \text{cosb}_\varepsilon w' \pm \text{cosb}_\varepsilon w \, \text{sinb}_\varepsilon w' ,$$
$$\text{cosb}_\varepsilon (w \pm w') = \text{cosb}_\varepsilon w \, \text{cosb}_\varepsilon w' \pm \varepsilon^{-2} \text{sinb}_\varepsilon w \, \text{cosb}_\varepsilon w' ,$$

from which others are easily obtained. These include the corresponding formulae in hyberbolic and circular functions as special cases, $b = e$ and $\varepsilon = 1$ giving the hyperbolic, $b = e$ and $\varepsilon = i = \sqrt{-1}$ the circular forms.

BERKELEY, *August*, 1891.

Corrected Seminvariant Tables for the Weights 11 & 12.

By Prof. Cayley.

The tables in my paper, "Seminvariant Tables," *A. M. J.*, t. VII (1885), pp. 59–73, are not in the best form, but the deviations present themselves only in a few columns of the tables for the weights 11 and 12, viz. in the former of these two columns, and in the latter a single column, ought to be replaced by linear combinations of other columns; there are besides columns which should be new named; in regard hereto there is a point of theory which requires to be made clear. I remark that in each table the literal terms are in alphabetical order (AO); this is the proper order for the final terms, and although (as about to be explained) the proper order for the initial terms is the counter order (CO), yet as the tables cannot be at the same time arranged in the one and the other order, I adhere to the AO as the proper arrangement for the terms of the tables; we have, however, to introduce the notion that in general it is not the top term of a column which is to be regarded as the initial term, and in connection herewith to consider how the columns are to be named. An instance first presents itself for the weight 11: we have, see column 5 of the table for that weight here given, a seminvariant

$$\begin{array}{ll}
fg & +\;\;1 \\
b^2j & \\
bci & \\
bdh & \\
beg & -\;\;5 \\
bf^2 & -\;\;6 \\
c^2h & \overset{\sim\sim\sim}{\underset{\sim\sim\sim}{-\;16}} \\
\vdots & \\
b^3e^2 & +\;70;
\end{array}$$

this is to be regarded, not as a seminvariant $fg - b^3e^2$, (it is hardly necessary to remark that, here and elsewhere the — is not a minus sign, but a mere stroke), but as a seminvariant $c^2h - b^3e^2$, viz. c^2h is a term entering into the seminvariant, and which, although it is in AO subsequent, it is in CO precedent to the terms fg, beg and bf^2. The seminvariant contains the term $-16c^2h$ and other terms with the letter h, and it is a misnomer to call it $fg - b^3e^2$, a name implying that the highest letter thereof is g. Instead of the stroke, it would perhaps be better to write ∞, for instance $c^2h \infty b^3e^2$, where of course ∞ would be used as a mere conventional symbol.

For greater clearness I give here the express definition of counter order, (CO), viz. whereas in AO we begin with the lowest letters, in CO we begin contrariwise with the highest letters. A term containing a higher letter or higher power of such letter precedes a term containing a lower letter or lower power of the same letter—or in the easiest form, the counter order is the alphabetical order corresponding to the reversed arrangement $z, y, \ldots f, e, d, c, b$ of the letters.

A symbol as above $c^2h - b^3e^2$ may be regarded as referring to a set of terms c^2h, b^3e^2 and all the terms which are in CO subsequent to c^2h and in AO precedent to b^3e^2: as by supposition the terms are arranged in AO the set includes no term lower than b^3e^2, or say the bottom term b^3e^2 is also the final term of the set, but it does include terms fg, beg and bf^2 higher than c^2h, and thus the top term fg is not, but c^2h is, the initial term of the set. It should be remarked that a seminvariant $ch - b^3e^2$ need not include all the terms of the set as just defined, there may very well be terms with a coefficient zero, or say accidental zeros: an instance presents itself, weight 10, where in the column $eg - bd^3$ we have $0ce^2$, no term in ce^2.

The changes actually required are very slight, viz.

Weight 11, instead of

$fg - b^3e^2$, we require $c^2h - b^3e^2$, old $fg - b^3e^2$, new named,
$c^2h - b^3d^2$, " $fg - b^3d^2$; linear combination $(fg - b^3e^2) + 8\,(c^2h - b^3d^2)$,
$de^2 - b^5d^2$, " $c^3f - b^5d^2$, old $de^2 - b^5d^2$, new named,
$c^3f - b^3c^4$, " $de^2 - b^3c^4$, linear combination $(de^2 - b^5d^2) + 6\,(c^3f - b^3c^4)$.

Weight 12, instead of

$cf^2 - bcd^3$, we require $d^2g - bcd^3$, old $cf^2 - bcd^3$, new named,
$d^2g - c^3d^2$, " $cf^2 - c^3d^2$, linear combination $(cf^2 - bcd^3) - 5\,(d^2g - c^3d^2)$;

but I have thought it desirable to give the complete tables for the weights in question, 11 and 12; and I have also rearranged the entire columns of the two tables so as to present in each of them the *finals* in AO. This is the case with the existing tables, except that in the table weight 10 there is a single transposition. Instead of the columns $cdf - b^4d^2$, $ce^2 - c^5$, we ought to have $ce^2 - c^5$, $cdf - b^4d^2$. The complete list up to the weight 12 is

$w =$	
2	$c - b^2$
3	$d - b^3$
4	$e - c^2$
	$c^2 - b^4$
5	$f - bc^2$
	$cd - b^5$
6	$g - d^2$
	$ce - c^3$
	$d^2 - b^2c^2$
	$c^3 - b^6$
7	$h - bd^2$
	$cf - bc^3$
	$de - b^3c^2$
	$c^2d - b^7$
8	$i - e^2$
	$cg - cd^2$
	$df - b^2d^2$
	$e^2 - c^4$
	$c^2e - b^2c^3$
	$cd^2 - b^4c^2$
	$c^4 - b^8$
9	$j - be^2$
	$ch - d^3$
	$dg - bcd^2$
	$ef - b^3d^2$
	$c^2f - bc^4$
	$cde - b^3c^3$
	$d^3 - b^5c^2$
	$c^3d - b^9$
10	$k - f^2$
	$ci - ce^2$
	$dh - b^2e^2$
	$eg - bd^3$
	$f^2 - c^2d^2$
	$c^2g - b^2cd^2$
	$ce^2 - c^5$
	$cdf - b^4d^2$
	$d^2e - b^2c^4$
	$c^3e - b^4c^3$
	$c^2d^2 - b^6c^2$
	$c^5 - b^{10}$
11	$l - bf^2$
	$cj - de^2$
	$di - bce^2$
	$eh - cd^3$
	$c^2h - b^3e^2$
	$fg - b^2d^3$
	$cef - bc^2d^2$
	$cdg - b^3cd^2$
	$d^2f - bc^5$
	$c^3f - b^5d^2$
	$de^2 - b^3c^4$
	$c^2de - b^5c^3$
	$cd^3 - b^7c^2$
	$c^4d - b^{11}$
12	$m - g^2$
	$ck - cf^2$
	$ei - e^3$
	$dj - b^2f^2$
	$fh - bde^2$
	$g^2 - c^2e^2$
	$ceg - d^4$
	$c^2i - b^2ce^2$
	$d^2g - bcd^3$
	$cf^2 - c^3d^2$
	$cdh - b^4e^2$
	$def - b^3d^3$
	$e^3 - b^2c^2d^2$
	$c^2e^2 - c^6$
	$c^3g - b^4cd^2$
	$cd^2e - b^2c^5$
	$c^2df - b^6d^2$
	$d^4 - b^4c^4$
	$c^4e - b^6c^3$
	$c^3d^2 - b^8c^2$
	$c^6 - b^{12}$

The two new tables are as follows: some accidental numerical errors have been corrected.

TABLE, WEIGHT 11.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	
		l	cj	di	eh	c^2h	fg	cef	cdg	d^2f	c^3f	de^2	c^2de	cd^3	c^4d	
	l	+ 1														
	bk	− 11														
	cj	+ 35	+ 2													
	di	− 75	− 9	+ 1												
	eh	+ 90	+ 14	− 2	+ 1											
	fg	− 42	− 7	+ 1	− 1	+ 1	+ 1									
	b^2j	+ 20	− 2													
	bci	− 90	+ 9	− 3												
	bdh	+ 240	+ 16		− 4											
10	beg	− 420	− 63	+ 9	− 2	− 5	− 5									10
	bf^2	+ 252	+ 42	− 6	+ 6	− 6	− 6									
	c^2h		− 30	+ 10	+ 3	− 16										
	cdg		+ 70	− 26	− 2	+ 58	+ 2		+ 1							
	cef		− 21	+ 7	− 6	+ 5	+ 45	+ 1	− 3							
	d^2f		− 56	+ 24	+ 10	− 100	− 100	− 3	+ 6	+ 1						
	de^2		+ 35	− 15	− 5	+ 60	+ 60	+ 2	− 4	− 1	+ 1	+ 1				
	b^3i			+ 2												
	b^2ch			− 8		+ 32										
	b^2dg			+ 8	+ 20	− 48	+ 8		− 1							
20	b^2ef			− 4	− 18	+ 40		− 1	+ 3							20
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	

TABLE, WEIGHT 11.—*Continued.*

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	
		l	cj	di	eh	c^2h	fg	cef	cdg	d^2f	c^3f	de^2	c^2de	cd^3	c^4d	
21	bc^2g			+ 8	− 15	− 62	− 6		− 3							21
	$bcdf$			− 16	− 24	+232	+408	+ 14	− 30	− 6						
	bce^2			+ 10	+ 45	−205	−405	− 11	+ 27	+ 3	− 3	− 3				
	bd^2e				− 10	+ 20	+ 20	− 1	+ 2	+ 3	− 8	− 8				
	c^3f				+ 27	− 54	−270	− 9	+ 27	+ 4	− 12					
	c^2de				− 45	+ 90	+450	+ 14	− 45	− 6	+ 36	+ 6	+ 1			
	cd^3				+ 20	− 40	−200	− 6	+ 20	+ 2	− 18		− 1	+ 1		
	b^4h					− 16										
	b^3cg					+ 56			+ 5							
30	b^3df					−112	−288	− 8	+ 18	+ 4						30
	b^3e^2					+ 70	+270	+ 9	− 23	− 2	+ 2	+ 2				
	b^2c^2f						+216	+ 6	− 27	− 3	+ 36					
	b^2cde						−360	− 16	+ 51	+ 6	− 36	+ 24	− 2			
	b^2d^3						+160	+ 8	− 24	− 8	+ 34	+ 16	+ 1	− 1		
	bc^3e							+ 3		− 2	− 48	− 18	− 3			
	bc^2d^2							− 2		+ 6	+ 18	− 24	+ 5	− 9		
	c^4d									− 1	+ 3	+ 9	− 1	+ 4	+ 1	
	b^5g								− 2							
	b^4cf								+ 6		− 36					
40	b^4de								− 2	− 4	+ 14	− 16	+ 1			40
	b^3c^2e								− 6	+ 3	+ 72	+ 12	+ 8			
	b^3cd^2								+ 4	+ 8	− 78	− 48	− 7	+ 15		
	b^2c^3d									− 10	+ 80	+ 72	− 5	+ 11	− 4	
	bc^5									+ 3	− 9	− 27	+ 3	− 12	− 3	
	b^6f										+ 12					
	b^5ce										− 80		− 7			
	b^5d^2										+ 20	+ 32	+ 2	− 6		
	b^4c^2d											− 48	+ 10	− 89	+ 6	
	b^3c^4											+ 18	− 5	+ 29	+ 14	
50	b^7e												+ 2			50
	b^6cd												− 4	+ 32	− 4	
	b^5c^3												+ 2	− 28	− 26	
	b^8d													− 8	+ 1	
	b^7c^2													+ 6	+ 24	
	b^9c														− 11	
56	b^{11}														+ 2	56
		bf^2	de^2	bce^2	cd^3	b^3e^2	b^2d^3	bc^2d^2	b^3cd^2	bc^5	b^5d^2	b^3c^4	b^5c^3	b^7c^2	b^{11}	
		±638	±188	±70	±132	±664	±1640	±57	±170	±43	±278	±192	±35	±98	±48	

TABLE, WEIGHT 12.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
	m	ck	ei	dj	fh	g^2	ceg	c^2i	d^2g	cf^2	cdh	def	e^3	c^2e^2	c^3g	cd^2e	c^2df	d^4	c^4e	c^3d^2	c^6
m	+1																				
bl	−12																				
ck	+66	+3																			
dj	−220	−15		+15																	
ei	+495	+40	+1	−40																	
fh	−792	−70	−4	+70	+25																
g^2	+462	+42	+3	−40	−24	+1															
b^2k		−3																			
bcj		+15		−45																	
bdi		−25	−4	+25																	
beh		+30	+12	−30	−125																
bfg		−14	−8	+12	+113	−12															
c^2i		−15	+3	+150				+1													
cdh		+40	−8	−400	+50			−4			+4										
ceg		−70	−22	+700	+680	−70	+1	+8			−8										
cf^2		+42	+24	−420	−675	+100	−1	−5	+2	+2	+5										
d^2g			+24		−570	+80	−1		+5												
def			−36		+925	−200	+2		−19	−4		+18									
e^3			+15		−400	+100	−1		+12	+2		−17	+1								
b^3j				+30																	
b^2ci				−135				−2													
b^2dh				+360	+200			+4			−4										
b^2eg				−630	−525	+100	−1	−8			+8										
b^2f^2				+378	+336	−64	+1	+5	−2	−2	−5										
bc^2h					−150			+4			−12										
$bcdg$					+350	−200	+2	−4	−30		+4										
$bcef$					−105	+20	−2	+2	+37	−8	−2	−54									
bd^2f					−280	+320	−2		+46	+16		−72									
bde^2					+175	−200	+2		−49	−4		+114	−12								
c^3g						+100	−1	−4	+20		+32				+1						
c^2df						−200	+2	+8	−49	−4	−64	+54			−3		+3				
c^2e^2						+125	+1	−5	−32	+28	+40	+162	−18	+1							
cd^2e							−3		+91	−44		−342	+54	−2	+4	+1					
d^4							+1		−32	+18		+135	−27	+1	−2	−1	−2	+1			
b^4i								+1													
b^3ch								−4			+20										
b^3dg								+4	+20		−4										
b^3ef								−2	−18	+12	+2	+36									
b^2c^2g								+4	−15		−60				−3						
b^2cdf								−8	−24	−24	+120	+216			+6		−6				
b^2ce^2								+5	+45	−30	−75	−360	+54	−2	+2						
b^2d^2e									−10	+20		+66	−6	+2	−6	−1					
bc^3f									+27	+12		−162			+3		−9				
bc^2de									−45	+60		+486	−180	+4	−8	−6	−15				
bcd^3									+20	−40		−252	+108	−4	+7	+8	+24	−12			
c^4e										−30		−81	+81	−2	+1	+4	+30		+1		
c^3d^2										+20		+54	−54	+2	−2	−5	−28	+8	−1	+1	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

TABLE, WEIGHT 12.—*Continued.*

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
		m	ck	ei	dj	fh	g^2	ceg	c^2i	d^2g	cf^2	cdh	def	e^3	c^2e^2	c^3g	cd^2e	c^2df	d^4	c^4e	c^3d^2	c^6
48	b^5h											−8										
	b^4cg											+28				+3						
50	b^4df											−56	−144			−3		+3				
	b^4e^2											+35	+135	−27	+1							
	b^3c^2f												+108			−6		+24				
	b^3cde												−180	+108	−4	+9	+10	+30				
	b^3d^3												+80	−64		+1	−4	−16	+8			
	b^2c^3e													−54	+2	+2	−7	−75		−4		
	$b^2c^2d^2$													+36	+4	−12	−9	+6	+30	+3	−3	
	bc^4d														−4	+8	+14	+48	−48	+2	−6	
	c^6														+1	−2		+2	+16	−1	+4	+1
	b^6g															−1	−4	−16				
60	b^5cf															+3		−21				
	b^5de															−1	−4	−15				
	b^4c^2e															−3	+3	+60		+6		
	b^4cd^2															+2	+8		−48	−3	+3	
	b^3c^3d																−10	−40	+68	−6	+22	
	b^2c^5																+3	+12	−24	+3	−15	−6
	b^7f																	+6				
	b^6ce																	−15		−4		
	b^6d^2																	+10	+16	−23	−1	
	b^5c^2d																		−24	+30	−30	
70	b^4c^4																		+9	−3	+21	+15
	b^8e																			+1		
	b^7cd																			+2	+14	−20
	b^6c^3																			−3	−9	
	b^9d																				−4	
	b^8c^2																				+3	+15
	$b^{10}c$																					−6
77	b^{12}																					+1
		g^2	cf^2	e^3	b^2f^2	bde^2	c^2e^2	d^4	b^2ce^2	bcd^3	c^3d^2	b^4e^2	b^3d^3	$b^2c^2d^2$	c^6	b^4cd^2	b^2c^5	b^6d^2	b^4c^4	b^6c^3	b^8c^2	b^{12}
		±1024	±212	±82	±1740	±2854	±946	±12	±46	±325	±190	±298	±1664	±442	±18	±52	±51	±258	±156	±48	±68	±32

Sur les fonctions à espaces lacunaires.

PAR H. POINCARÉ.

M. Weierstrass dans un mémoire intitulé 'Zur Funktionenlehre' et inséré dans les Berliner Monatsberichte a appelé l'attention des géomètres sur certaines fonctions présentant des singularités spéciales. Au lieu de présenter un nombre fini ou infini de points singuliers essentiels *isolés* elles offrent des lignes singulières essentielles ou même des *espaces lacunaires* à l'intérieur desquels elles cessent d'exister. Dans une lettre à M. Mittag-Leffler, insérée dans les Acta Societatis Scientiarum Fennicæ M. Hermite a retrouvé les mêmes résultats par une voie toute différente. D'après les conseils de M. Hermite j'ai entrepris de rechercher de nouveaux exemples de la particularité signalée par les deux savants géomètres.

Il y a une infinité de manières de définir une fonction, et si on ne s'imposait a priori aucune condition, rien ne serait plus facile que de concevoir une transcendante présentant un espace lacunaire quelconque; on pourrait imaginer par exemple une fonction définie de la manière suivante; elle devrait être égale à 1 à l'extérieur d'un certain cercle, et cesser d'exister à l'intérieur de ce cercle. Ce cercle serait alors un *espace lacunaire.* Si donc on donnait au mot, *fonctions à espaces lacunaires* le sens étendu qu'il semble comporter d'abord, on pourrait en imaginer arbitrairement une infinité. Il est donc nécessaire de préciser ce qu'on doit entendre par cette expression, *fonctions à espaces lacunaires.* C'est ce qui est facile, grâce à une conception nouvelle des fonctions analytiques qui a son origine dans les travaux de Cauchy et que M. Weierstrass a si clairement exposée dans son mémoire 'Zur Functionenlehre' (Monatsberichte, Août 1880, page 12).

Considérons une série développée suivant les puissances croissantes de $x-x_0$. Elle sera convergente à l'intérieur d'un certain cercle C_0 ayant pour centre x_0 et pour rayon R. Si on ne s'occupait que du développement lui-même, on pourrait considérer la fonction définie par la série comme cessant d'exister à l'extérieur du cercle de convergence, et toute la région du plan extérieure à ce cercle comme formant un espace lacunaire. Ainsi comprise, la fonction à espaces lacunaires ne serait pas une notion analytique nouvelle. Mais il est un moyen bien connu d'étendre au-delà du cercle de convergence le domaine où la fonction envisagée existe. Si l'on considère un point x_1 intérieur au cercle de convergence, on pourra par la formule de Taylor développer la fonction en série ordonnée suivant les puissances de $x - x_1$ et convergente à l'intérieur d'un certain cercle C_1. A l'intérieur de C_1, on prendra un point x_2 et on developpera la fonction en série ordonnée suivant les puissances de $x - x_2$ et convergente à l'intérieur d'un certain cercle C_2. La fonction se trouvera alors définie non seulement à l'intérieur du premier cercle de convergence, mais l'intérieur de C_1, de C_2, etc.

Pour la plupart des fonctions qui ont été jusqu'ici l'objet des travaux des géomètres, les cercles tels que C_1, C_2, etc., recouvrent tout le plan, soit une fois, soit plusieurs fois, soit une infinité de fois, en laissant seulement de côté certains points isolés, appelés points singuliers. La fonction existe partout, sauf en des points isolés. *Il n'y a pas d'espace lacunaire.*

Mais il n'en est pas toujours ainsi; il peut arriver que les cercles C_1, C_2, etc., laissent de côté non des points isolés, mais toute une ligne, ou même toute une région du plan. M. Weierstrass a le premier mis cette vérité en lumière, et après lui M. Hermite a défini à l'aide d'intégrales multiples définies des transcendantes qui n'ont d'existence que dans un domaine limité.

On pourrait citer un grand nombre d'autres exemples de ce fait analytique. Ainsi l'on sait que les fonctions définies par les séries:

$$1 + \frac{1}{2}x^3 + \frac{1}{2^2}x^3 + \dots\dots + \frac{1}{2^n}x^{3^n} + \dots\dots$$

et

$$x\phi(1) + x^2\phi(2) + \dots\dots + x^n\phi(n) + \dots\dots$$

(où $\phi(n)$ représente la somme des puissances p^{es} des diviseurs de n) n'existent qu'à l'intérieur du cercle qui a pour centre l'origine et pour rayon l'unité. Il en est de même de certaines fonctions que j'ai appelées fuchsiennes.

Les exemples que je veux étudier spécialement dans la présente note presenteront les particularités suivantes. Le plan sera divisé en deux régions, l'une extérieure, l'autre intérieure à un certain contour C. A l'extérieur du contour la fonction envisagée sera holomorphe et uniforme (et par conséquent finie, continue, monodrome et monogène). A l'intérieur du contour elle cessera d'exister. La région intérieure à C sera un espace lacunaire.

Je supposerai dans ce qui va suivre que la ligne qui limite C ait en chaque point une tangente et un rayon de courbure afin qu'on puisse construire un cercle tangent à cette ligne, ayant son centre en un point quelconque de la partie du plan qui est en dehors de C et de telle façon que ce cercle soit tout entier extérieur à C.

Si x_0 est un point quelconque extérieur à C la fonction sera développable suivant les puissances de $x - x_0$; le cercle de convergence sera tangent extérieurement à C. Réciproquement si (x_0 étant un point quelconque extérieur à C) une fonction est développable suivant les puissances de $x - x_0$, de telle sorte que le cercle de convergence soit tangent extérieurement à C, il est clair que cette fonction offrira un espace lacunaire qui sera la région intérieure au contour C.

Voici maintenant comment je définirai une transcendante jouissant de ces propriétés. Envisageons la série suivante:

$$\sum_{n=0}^{n=\infty} \frac{A_n}{x - b_n} = \phi(x). \tag{1}$$

Je suppose:

1° que la série:

$$\sum_{n=0}^{n=\infty} A_n \tag{2}$$

soit absolument convergente.

2° que tous les points b_n soient intérieurs à C ou sur le contour C lui-même.

3° que si l'on prend sur le contour C un arc quelconque et aussi petit que l'on voudra, il y ait toujours une infinité de points b_n sur cet arc.

Je pose:

$$R_p = \sum_{n=p}^{n=\infty} |A_n|, \qquad S = \sum_{n=0}^{n=\infty} |A_n|.$$

La série (2) étant absolument convergente, on pourra prendre p assez grand pour que R_p soit aussi petit que l'on veut.

Je dis d'abord que si x_0 est extérieur à C, la fonction $\phi(x)$ définie par la série (1) peut se développer en série suivant les puissances de $x - x_0$, et que cette série est convergente à l'intérieur du cercle qui a pour centre x_0 et qui est tangent extérieurement à C. Si en effet R est le rayon de ce cercle, on aura pour tous les points b_n,

$$|b_n - x_0| > R. \qquad \text{Posons } |x - x_0| = \Theta . R.$$

Supposons que x soit intérieur au cercle qui a pour centre x_0 et pour rayon R, on aura: $\Theta < 1.$

On aura évidemment:

$$-\phi(x) = \sum_{n=0}^{n=\infty}\Big[\sum_{q=0}^{q=\infty}\Big(A_n \frac{(x-x_0)^q}{(b_n-x_0)^{q+1}}\Big)\Big].$$

Il est clair:

1° que la série à termes positifs et à double entrée

$$\sum_{n=0,\ q=0}^{n=\infty,\ q=\infty} \frac{A_n \Theta^{q+1}}{(x-x_0)} \tag{3}$$

est absolument convergente.

2° que

$$\operatorname{mod}\Big[A_n \frac{(x-x_0)^q}{(b_n-x_0)^{q+1}}\Big] < \frac{\operatorname{mod} A_n \Theta^{q+1}}{\operatorname{mod}(x-x_0)}.$$

Il en résulte que la série à double entrée:

$$\sum_{n=0,\ q=0}^{n=\infty,\ q=\infty} A_n \frac{(x-x_0)^q}{(b_n-x_0)^{q+1}} \tag{4}$$

est absolument convergente et que sa somme est indépendante de l'ordre des termes.

La somme de la série (4) sera donc $-\phi(x)$ quel que soit l'ordre des termes. On aura donc:

$$-\phi(x) = \sum_{q=0}^{q=\infty} B_q (x-x_0)^q \tag{5}$$

en posant:

$$B_q = \sum_{n=0}^{n=\infty} A_n (b_n - x_0)^{-(q+1)}.$$

J'ai donc démontré à la fois:

1° que si x est extérieur à C la série (2) est convergente et la fonction $\phi(x)$ qu'elle définit est holomorphe et uniforme.

2° que si x est intérieur au cercle qui a pour centre x_0 et pour rayon R et qui est tangent extérieurement à C, la série (5) est convergente.

Je dis maintenant que la série (5) est divergente si x est sur ce cercle ou extérieur à ce cercle et pour le démontrer, je suppose d'abord que x_0 soit sur la normale élevée à C en un des points b_n, au point b_n par exemple.

Je me propose de faire voir que le terme

$$B_q R^q$$

ne tend pas vers 0 quand q tend vers l'infini; je vais montrer en effet que l'on peut prendre q assez grand pour que:

$$\operatorname{mod} R^q \left[B_q - A_k (b_k - x_0)^{-(q+1)}\right] < \varepsilon,$$

quelque petit que soit ε.

Soit p un nombre entier assez grand pour que:

$$R_p < \frac{\varepsilon}{2} R.$$

Supposons en même temps

$$p > k.$$

Décrivons du point x_0 comme centre un cercle de rayon R' plus grand que R, mais assez petit pour que tous les points:

$$b_0,\ b_1,\ b_2,\ \ldots\ b_{k-2},\ b_{k-1},\ b_{k+1},\ b_{k+2},\ \ldots\ b_p$$

soient extérieurs à ce cercle. On aura:

$$\frac{R}{R'} < 1.$$

Soit maintenant q un nombre entier assez grand pour que:

$$\frac{S}{R'} \left(\frac{R}{R'}\right)^q < \frac{\varepsilon}{2}.$$

On aura:

$$B_q - A_k (b_k - x_0)^{-(q+1)} = \sum_{n=0}^{n=k-1} \frac{A_n}{(b_n - x_0)^{q+1}} + \sum_{n=k+1}^{n=p-1} \frac{A_n}{(b_n - x_0)^{q+1}} + \sum_{n=p}^{n=\infty} \frac{A_n}{(b_n - x_0)^{q+1}}.$$

On aura:

$$\operatorname{mod} R^q \left(B_q - A_k (b_k - x_0)^{-(q+1)}\right) < \operatorname{mod} \left[\sum_{n=0}^{n=k-1} \frac{A_n R^q}{(b_n - x_0)^{q+1}} + \sum_{n=k+1}^{n=p-1} \frac{A_n R^q}{(b_n - x_0)^{q+1}}\right]$$

$$+ \operatorname{mod} \sum_{n=p}^{n=\infty} \frac{A_n R^q}{(b_n - x_0)^{q+1}} < \sum_{n=0}^{n=k-1} \frac{\operatorname{mod} A_n}{R'} \left(\frac{R}{R'}\right)^q$$

$$+ \sum_{n=k+1}^{n=p-1} \frac{\operatorname{mod} A_n}{R'} \left(\frac{R}{R'}\right)^q + \sum_{n=p}^{n=\infty} \frac{\operatorname{mod} A_n}{R} < \frac{S}{R'} \left(\frac{R}{R'}\right)^q + \frac{R_p}{R} < \varepsilon.$$

On a donc:

$$\lim R^q \left(B_q - A_k (b_k - x_0)^{-q+1}\right) = 0.$$

Or:

$$\operatorname{mod}\left(R^q A_k (b_k - x_0)^{-q+1}\right) = \frac{\operatorname{mod} A_k}{R}.$$

Il est donc impossible que $R^q A_k (b_k - x_0)^{-q+1}$ et par conséquent que $R^q B_q$ tende vers 0.

Supposons maintenant que x_0 ne soit pas sur la normale élevée à C en l'un des points b_n; je dis que la série (6) est encore divergente quand:

$$\operatorname{mod}(x - x_0) > R.$$

En effet supposons que cela ne soit pas vrai et que le cercle de convergence ait un rayon R' plus grand que R. Ce cercle de convergence découperait sur le contour C un certain arc sur lequel, par hypothèse, il devrait y avoir une infinité de points b_n. Soit b_k l'un de ces points. Elevons en ce point une normale à C et prenons sur cette normale un point x_1 assez voisin de b_k pour que le cercle K qui passe par b_k et qui a x_1 pour centre soit tout entier intérieur au cercle qui a pour rayon R' et pour centre x_0; cela est évidemment toujours possible. La fonction $\phi(x)$ pourrait alors se développer en série suivant les puissances de $x - x_1$ et cette série devrait être convergente, non seulement à l'intérieur du cercle K, mais sur la circonférence de ce cercle, ce qui est contraire à ce que je viens de démontrer.

Il est donc démontré que le cercle de convergence de la série (6) est toujours tangent extérieurement à C.

Donc la fonction $\phi(x)$ est holomorphe et uniforme à l'extérieur de C et présente un espace lacunaire à l'intérieur de ce contour.

Je vais maintenant citer quelques exemples de séries satisfaisant aux conditions imposées à la série (1).

Soit d'abord:

$$\phi(x) = \sum \frac{u_1^{m_1} u_2^{m_2} \dots u_p^{m_p}}{x - \dfrac{m_1\alpha_1 + m_2\alpha_2 + \dots + m_p\alpha_p}{m_1 + m_2 + \dots + m_p}}. \tag{6}$$

Je suppose:

1° que $u_1, u_2, \dots u_p$ sont des quantités données de module plus petit que 1.

2° que $\alpha_1, \alpha_2, \dots \alpha_p$ sont des constantes quelconques.

3° que $m_1, m_2, \dots m_p$ prennent sous le signe Σ tous les systèmes de valeurs entières positives.

J'envisage le polygône P défini par les conditions suivantes:

1° Il est convexe.

2° Tous ses sommets font partie du système des points $\alpha_1, \alpha_2 \ldots . \alpha_p$.

3° Tous les points $\alpha_1, \alpha_2 \ldots . \alpha_p$ qui ne sont pas des sommets du polygône P sont sur le périmètre de ce polygône ou à son intérieur.

Il est clair que:

1° La série

$$\sum \operatorname{mod}\left(u_1^{m_1} u_2^{m_2} \ldots . u_p^{m_p}\right)$$

est convergente.

2° Tous les points

$$\frac{m_1\alpha_1 + m_2\alpha_2 + \ldots . + m_p\alpha_p}{m_1 + m_2 + \ldots . + m_p}$$

sont sur le périmètre de P ou bien à l'intérieur de ce polygône.

3° Sur tout segment, si petit qu'il soit, de l'un des côtés de P, il y a une infinité de points:

$$\frac{m_1\alpha_1 + m_2\alpha_2 + \ldots . + m_p\alpha_p}{m_1 + m_2 + \ldots . + m_p}.$$

Soit en effet $\alpha_1 \alpha_2$ le côté du polygône consideré, il est clair qu'on pourra choisir les entiers positifs m_1 et m_2 (et cela d'une infinité de manières) de telle sorte que le point

$$\frac{m_1\alpha_1 + m_2\alpha_2}{m_1 + m_2}$$

soit situé sur un segment donné du côté $\alpha_1\alpha_2$.

Il en résulte que la fonction $\phi(x)$ est holomorphe et uniforme à l'extérieur de P et présente un espace lacunaire à l'intérieur de ce polygône.

Dans le cas où $p = 3$, l'espace lacunaire se réduit au triangle $\alpha_1\alpha_2\alpha_3$.

Dans le cas où $p = 2$, l'espace lacunaire se réduit à une ligne singulière essentielle qui est le segment de droite $\alpha_1\alpha_2$.

Comme second exemple je citerai la fonction dont voici l'origine.

Soit l'equation aux différences partielles:

$$u_1F_1\frac{dz}{du_1} + u_2F_2\frac{dz}{du_2} + \ldots . + u_nF_n\frac{dz}{du_n} = z. \tag{7}$$

$F_1, F_2, \ldots F_n$ sont des fonctions des n variables $u_1, u_2 \ldots u_n$ et du paramètre x, holomorphes pour toutes les valeurs de x et lorsque les modules de $u_1, u_2 \ldots u_n$ sont suffisamment petits. Elles se réduisent respectivement à

$$1, \frac{x-\alpha_2}{x-\alpha_1}, \ldots\ldots \frac{x-\alpha_n}{x-\alpha_1}$$

quand on y annule tous les u.

Dans une thèse que j'ai soutenue devant la Faculté des Sciences de Paris le 1er Août 1879, j'ai démontré que si le point x est extérieur au polygône convexe P circonscrit aux n points $\alpha_1, \alpha_2 \ldots \alpha_n$, il existe une série S ordonnée suivant les puissances des u, convergente et satisfaisant à l'équation (7) pourvu que les modules de ces variables soient assez petits. Les coëfficients de cette série sont des fonctions rationnelles de x; si on donne aux u des valeurs de module suffisamment petit et qu'on les considère comme des constantes, la somme de la série est une fonction de x, et l'on peut voir qu'elle est analogue à la fonction $\phi(x)$ définie par la série (1) et qu'elle présente comme elle un espace lacunaire. Le polygône P est compris tout entier dans cet espace lacunaire.

On remarquera que dans la démonstration qui précède, il y a un procédé qui joue un rôle essentiel. On décrit du point x_0 comme centre un cercle avec un rayon R' plus grand que

$$R = |x_0 - b_k|$$

et cependant assez petit pour que tous les points

$$b_0, b_1, b_2, \ldots, b_{k-2}, b_{k-1}, b_{k+1}, b_{k+2}, \ldots b_p,$$

c'est à dire les points b_n tels que $n \leqq p$, $n \gtrless k$ soient tous extérieurs à ce cercle, et que

$$\sum_{n=p}^{n=\infty} |A_n| < \frac{\varepsilon}{2} R,$$

la sommation étant ainsi étendue à tous les indices $n > p$.

Les points b_n sont ainsi répartés en deux catégories:

1° ceux pour lesquels $n \leqq p$; *ils sont en nombre fini* et ils sont tous extérieurs au cercle de rayon R' à l'exception d'un seul, le point b_k.

2° ceux pour lesquels $n > p$ qui sont en nombre infini, mais si p est assez grand la somme des modules des coëfficients correspondants A_n sera aussi petite qu'on voudra.

C'est sur la possibilité de cette répartition que repose toute la démonstration et c'est pour cette raison qu'elle n'est pas susceptible de diverses géneralisations que l'on croirait d'abord possibles.

Soit par exemple dans le plan une courbe fermée C et z un point mobile assujetti à rester sur cette courbe; soit x un point extérieur à la courbe et $f(z)$ une fonction quelconque de z. L'intégrale

$$\int \frac{f(z)dz}{z-x}$$

étendue à la courbe fermée C définit une fonction $\phi(x)$ de x. On pourrait croire que les raisonnements qui précèdent lui sont applicables, à condition que l'intégrale

$$\int |f(z)dz|$$

soit finie, et que la fonction $\phi(x)$ admet l'intérieur de C comme espace lacunaire. Il n'en est rien à cause de l'impossibilité de la répartition dont nous venons de parler. Il est vrai que cette fonction $\phi(x)$ reste holomorphe à l'extérieur de C, mais non pas que si x_0 est extérieur à C, le cercle de convergence relatif au développement de $\phi(x)$ suivant les puissances croissantes de $x-x_0$ soit tout entier à l'extérieur de C. Il suffit pour s'en convaincre de se rappeler que si $f(x)$ est une fonction *quelconque* holomorphe à l'extérieur de C et tendant vers 0 quand le point x s'éloigne indéfiniment, l'intégrale

$$\int \frac{f(z)dz}{z-x}$$

prise le long de C est précisément égale à $2i\pi f(x)$.

De même si le point z n'est plus assujetti à rester sur la courbe C elle même mais peut prendre une position quelconque à l'intérieur de cette courbe, si $f(z)$ est une fonction continue quelconque de z et $d\omega$ un élément de l'aire limitée par cette courbe, et que z désigne précisément l'affixe du centre de gravité de $d\omega$, l'intégrale

$$\int \frac{f(z)d\omega}{z-x}$$

étendue à l'aire limitée par C représentera une fonction $\phi(x)$ qui sera holomorphe à l'extérieur de C mais qui n'admettra pas en général la région intérieure à C comme espace lacunaire.

On obtient des resultats analogues dans la théorie du potentiel newtonien.

Soient M_1, M_2, ... M_n, ... un nombre infini de points dont les masses m_1, m_2, ... m_n soient toutes positives. Supposons que la série

$$\sum m_n$$

soit convergente; que tous ces points attirent un point mobile P de coördonnées x, y, et z conformément à la loi de Newton; que tous ces points soient à l'intérieur d'une certaine région R limitée par une surface S; et enfin que sur chaque élément si petit qu'il soit de cette surface S il y ait une infinité de ces points.

Soit alors $V(x, y, z)$ le potentiel de ces points attirants. On verrait alors par un raisonnement tout pareil à celui qui précède, que la fonction V est holomorphe à l'extérieur de R et qu'elle admet cette région R comme espace lacunaire.

Supposons au contraire que nous ayons affaire non pas à des points attirants discrets quoique en nombre infini et infiniment rapprochés les unes des autres, mais à une surface attirante, ou à un volume attirant, il n'en sera plus de même.

Si on considère par exemple un volume attirant limité par une surface S sans point singulier et que la dense soit une fonction holomorphe de x, y et z, la fonction V pourra être prolongée par continuation analytique à l'intérieur du volume attirant.

Supposons en particulier une sphère attirante homogène ayant son centre à l'origine, ayant pour rayon R et pour masse M. Alors on soit que la fonction

$$V = \frac{M}{\sqrt{x^2 + y^2 + z^2}}$$

loin d'admettre comme espace lacunaire l'intérieur de la sphère

$$x^2 + y^2 + z^2 = R^2$$

peut être prolongée par continuation analytique jusqu'au centre même de cette sphère.

Il arrive quelquefois qu'on a à envisager des développements en séries de la forme suivante:

$$\phi(x) = \Sigma R_n(x),$$

$R_n(x)$ étant rationnelle en x.

Supposons que ce développement soit convergent à l'intérieur d'une certaine courbe C, convergent également à l'extérieur de cette courbe, mais qu'il diverge pour tous les points de la courbe elle-même. Les développements de cette forme ont été étudiés par M. Weierstrass dans le mémoire que j'ai cité plus haut, ainsi que les diverses circonstances que je vais signaler.

Soit $\phi_1(x)$ la somme de la série à l'intérieur de la courbe C, $\phi_2(x)$ la somme de cette même série à l'extérieur de la courbe C. Il peut se faire d'abord que les fonctions $\phi_1(x)$ et $\phi_2(x)$ puissent être prolongées, par le procédé de la continuation analytique exposé au début du présent travail, la première à l'extérieur de C, la seconde à l'intérieur de C.

C'est ainsi que par exemple la série de M. Tannery:

$$\sum_{n=1}^{n=\infty}\left[\frac{x^n-1}{x^n+1}-\frac{x^{n-1}-1}{x^{n-1}+1}\right]$$

a pour somme $+1$ à l'extérieur du cercle:

$$|x|=1$$

et -1 à l'intérieur de ce cercle. Il est clair alors que les fonctions $+1$ et -1 peuvent être prolongées dans tout le plan.

Mais il peut arriver aussi que les fonctions ϕ_1 et ϕ_2 admettent comme espace lacunaire, la première l'extérieur de C, la seconde l'intérieur de C. M. Weierstrass cite des exemples de ce fait dans son mémoire et d'ailleurs plusieurs des développements en séries de la fonction modulaire présentent la même particularité.

Une question se pose alors. Nous avons:

$$\phi_1(x)=\Sigma R_n(x)$$

à l'intérieur de C et

$$\phi_2(x)=\Sigma R_n(x)$$

à l'extérieur de C. Avons-nous le droit de dire que la fonction ϕ_1 cesse d'exister à l'extérieur de C et la fonction ϕ_2 à l'intérieur, ou bien ne devrons-nous pas plutôt considérer les deux fonctions ϕ_1 et ϕ_2 comme le *prolongement naturel* l'une de l'autre?

On en serait d'abord tenté; mais on renoncera à cette manière de voir si l'on réfléchit qu'à ce point de vue, une fonction à espace lacunaire aurait dans cet espace une infinité de prolongements naturels possibles. C'est ce dont on peut se rendre compte par deux raisonnements différents que je vais appliquer à des courbes C particulières mais qu'on pourrait étendre, mutatis mutandis, à des courbes C quelconques.

Supposons d'abord que la courbe C soit un cercle.

Soit encore:

$$\phi_1(x) = \Sigma R_n(x), \quad \phi_2(x) = \Sigma R_n(x)$$

la première égalité ayant lieu à l'intérieur du cercle, la seconde à l'extérieur. Supposons que les deux fonctions ϕ_1 et ϕ_2 ne puissent être prolongées analytiquement au delà du cercle.

J'ai démontré l'existence de certaines fonctions que j'ai appelées fuchsiennes et têtafuchsiennes qui n'existent qu'à l'intérieur du cercle C et pour lesquelles par conséquent la région extérieure à ce cercle est un espace lacunaire. Les fonctions têtafuchsiennes sont susceptibles d'un développement dont les termes sont des fonctions rationnelles de x; je me suis longuement étendu sur ces développements dans mon mémoire sur les fonctions fuchsiennes (Acta Mathematica, tome 1).

Soit:

$$\sum H_n(x) = \Sigma H\left(\frac{\alpha_n x + \beta_n}{\gamma_n x + \delta_n}\right)(\gamma_n x + \delta_n)^{-2m}$$

un de ces développements. Il représentera en général à l'intérieur de C une fonction têtafuchsienne qui cessera d'exister à l'extérieur de ce cercle, et à l'extérieur de C il représentera une *autre* fonction têtafuchsienne qui cessera à son tous d'exister à l'intérieur de ce cercle.

Dans ce développement $H(x)$ est une fonction rationnelle quelconque, m est une entier et $\left(x, \dfrac{\alpha_n x + \beta_n}{\gamma_n x + \delta_n}\right)$ une substitution quelconque du groupe fuchsien.

Si tous les infinis de $H(x)$ sont à l'extérieur de C, la fonction têtafuchsienne représentée par notre développement à l'intérieur de C n'aura pas d'infinis et, aussi que je l'ai montré dans le mémoire cité, on peut d'une infinité de manières choisir H de telle sorte que cette fonction soit identiquement nulle. On aura alors:

$$\Sigma H_n(x) = 0 \quad \text{à l'intérieur de } C,$$
$$\Sigma H_n(x) = \Theta(x) \text{ à l'extérieur de } C;$$

Θ étant une fonction têtafuchsienne.

Quel serait alors le prolongement naturel de $\phi_1(x)$ à l'extérieur de C. Comme

$$\phi_1 = \Sigma R_n,$$

ce prolongement serait la somme de ce développement à l'extérieur de C, c'est à dire ϕ_2.

Mais on a aussi à l'intérieur de C

$$\phi_1 = \Sigma(R_n + H_n) \text{ puisque } \Sigma H_n = 0.$$

Le prolongement naturel de ϕ_1 à l'extérieur de C devrait encore être la somme de ce développement $\Sigma(R_n + H_n)$, c'est à dire $\phi_2 + \Theta$.

Ainsi une fonction à espace lacunaire serait susceptible de plusieurs "prolongements naturels"; c'est assez dire qu'il n'y en a aucun qui mérite ce nom.

Mais on peut s'en rendre compte encore d'une autre manière.

Considérons le plan des x comme divisé en deux parties par l'axe des quantités réelles. Soit $f(x)$ une fonction n'existant que dans la partie supérieure et étant partout holomorphe dans cette partie; soit $f_1(x)$ une fonction n'existant que dans la partie inférieure du plan et étant partout holomorphe dans cette partie. La moitié inférieure du plan est pour $f(x)$, la moitié supérieure pour $f_1(x)$, un espace lacunaire.

Je dis alors que je pourrai trouver deux fonctions $\phi(x)$ et $\psi(x)$ jouissant des propriétés suivantes:

1° Elles existeront dans tout le plan, sauf le long de certaines "coupures."

2° On aura $\phi + \psi = f$ dans moitié supérieure du plan.

3° On aura $\phi + \psi = f_1$ dans la moitié inférieure.

4° La fonction ϕ admettra pour coupure le segment de l'axe des quantités réelles compris entre les points $x = -1$ et $x = +1$.

5° La fonction ψ admettra pour coupure les deux autres segments de l'axe des quantités réelles, c'est à dire les segments $(-\infty, -1)$ et $(+1, +\infty)$.

S'il en est ainsi il est clair que si la fonction f avait un "prolongement naturel" dans la moitié inférieure du plan, ce prolongement ne pourrait être que f_1; car les fonctions ϕ et ψ existent dans tout le plan et le prolongement naturel de f devrait être comme la fonction f elle-même égal à la somme $\phi + \psi$. Mais f_1 est une fonction *quelconque* assujettie seulement à n'exister que dans la moitié inférieure du plan. Une fonction quelconque pourrait donc être regardée comme le prolongement naturel de f.

Pour démontrer le théorème que je viens d'énoncer, j'ai besoin d'abord d'établir le lemme suivant.

Soit $F(x)$ une fonction que je n'assujettis pas à être analytique, mais qui doit rester finie et continue pour toutes les valeurs *réelles et positives* de x, tout en croissant indéfiniment avec x.

Je dis alors que quelle que soit cette fonction $F(x)$, on pourra toujours trouver une fonction analytique entière $G(x)$ qui croisse assez rapidement pour que le rapport:

$$\frac{F(x)}{x\,G(x)}$$

tende vers 0 quand x croît indéfiniment par valeurs réelles positives.

En effet, on peut toujours trouver un nombre positif A_n assez grand pour que la plus grande valeur que prenne le module de $F(x)$ quand x varie de n à $n+1$ soit plus petite que A_n.

On aura alors (pour $n < x < n+1$):

$$\left|F(x)\right| < A_n < A_n\left(\frac{x}{n}\right)^{\lambda_n}. \tag{8}$$

Je puis supposer que λ_n est un entier positif plus grand que n et assez grand d'ailleurs pour que:

$$A_n < \left(\frac{n}{n-1}\right)^{\lambda_n}. \tag{9}$$

Considérons alors la série:

$$G(x) = \sum_{n=1}^{n=\infty} A_n\left(\frac{x}{n}\right)^{\lambda_n}.$$

Je dis d'abord que cette série converge pour toutes les valeurs de x et représente par conséquent une fonction entière. Il vient en effet:

$$\sqrt[n]{A_n\left(\frac{x}{n}\right)^{\lambda_n}} = A_n^{\frac{1}{n}}\left(\frac{x}{n}\right)^{\frac{\lambda_n}{n}} < \left(\frac{x}{n-1}\right)^{\frac{\lambda_n}{n}}.$$

Quand n croît indéfiniment $\frac{x}{n-1}$ tend vers 0 et comme $\frac{\lambda_n}{n}$ est plus grand que 1 il en sera de même de $\left(\frac{x}{n-1}\right)^{\frac{\lambda_n}{n}}$.

La série est donc convergente.

Je dis ensuite que pour les valeurs réelles et positives de x, on a

$$|F(x)| < G(x).$$

En effet tous les termes de la série $G(x)$ sont positifs, et l'inégalité (8) prouve qu'il y a toujours un de ces termes qui est plus grand que $|F(x)|$.

Il est clair alors que

$$\lim \frac{F(x)}{xG(x)} = 0 \quad (\text{pour } x = +\infty). \qquad \text{C. Q. F. D.}$$

On peut tirer de ce lemme divers corollaires. Comme $G(x)$ est une fonction entière dont les termes sont tous positifs et contiennent des puissances de x dont l'exposant dépasse toute limite, on aura quand x croîtra indéfiniment par valeurs réelles positives:

$$\lim \frac{x^m}{G(x)} = 0, \quad \lim \frac{G^m(x)}{e^{G(x)}} = 0$$

et par conséquent

$$\lim F(x)e^{-G(x)} = 0.$$

Je dis maintenant que si $F(x)$ est finie pour les valeurs réelles de x tant positives que négatives et suffisamment grandes en valeur absolue, on pourra trouver une fonction entière $G(x)$ telle que

$$\lim F(x)e^{-G(x)} = 0$$

quand x croît indéfiniment par valeurs *réelles* soit positives, soit négatives.

Soit en effet $F_1(x^2)$ la plus grande des deux quantités $|F(x)|$ et $|F(-x)|$ ce sera évidemment par définition même une fonction paire de x, c'est à dire ne changeant pas quand on change x en $-x$.

On pourra alors d'après le lemme qui précède trouver une fonction $G_1(x^2)$ telle que

$$\lim F_1(x^2)e^{-G_1(x^2)} = 0$$

pour $x^2 = +\infty$ (x^2 réel positif).

Si alors $G_1(x^2) = G(x)$ on aura

$$\lim F(x)e^{-G(x)} = 0$$

pour $x = \pm\infty$ (x réel, positif ou négatif).

Comme les valeurs que prend $F(x)$ quand $|x|$ est suffisamment grand, influent évidemment seules sur cette limite, il suffit pour que le lemme soit vrai que $F(x)$ soit fini pour les valeurs réeles de x suffisamment grandes, il suffit par exemple que $F(x)$ ne devienne infini que pour un nombre fini de valeurs réelles de x.

Considérons maintenant la circonférence

$$|x| = 1$$

décrite sur le segment $(-1, +1)$ comme diamètre et supposons que sur cette circonférence que j'appellerai C pour abréger la fonction $F(x)$ n'ait qu'un nombre fini d'infinis parmi lesquels le point $x = +1$.

Je dis qu'on pourra trouver une fonction entière $G(x)$ telle que

$$\lim F(x)e^{-G\left(i\frac{x+1}{x-1}\right)} = 0$$

quand x tendra vers 1 en suivant la circonférence.

Posons en effet

$$y = i\frac{x+1}{x-1}.$$

Si x est sur la circonférence C, y sera réel; si x tend vers 1 en suivant l'une des moitiés de la circonférence, y tend vers $+\infty$; si x tend vers 1 en suivant l'autre moitié, y tend vers $-\infty$. Si nous posons $F(x) = F_1(y)$, la fonction F_1 n'admettra qu'un nombre fini d'infinis *réels*. Alors on pourra trouver une fonction $G(y)$ telle que

$$\lim F_1(y)e^{-G(y)} = 0 \qquad \text{(pour } \lim y = \pm\infty)$$

on en déduira par conséquent:

$$\lim F(x)e^{-G(i\frac{x+1}{x-1}} = 0. \qquad \text{C. Q. F. D.}$$

De même, alors même que $F(x)$ deviendrait infinie pour $x = -1$, on pourra trouver une fonction entière $G'(x)$ telle que:

$$\lim F(x)e^{-G'\left(i\frac{x-1}{x+1}\right)}$$

quand x tend vers -1 en suivant la circonférence.

Ces corollaires établis venons à la question qui nous occupe.

Nous désignerons par $F(x)$ une fonction qui sera égale à $f(x)$ dans la moitié supérieure du plan et à $f_1(x)$ dans la moitié inférieure. Sur la circonférence que j'ai appelée C, la fonction $F(x)$ ainsi définie ne pourra avoir que deux infinis, $x = +1$ et $x = -1$.

On pourra alors construire les deux fonctions entières:

$$G\left(i\frac{x+1}{x-1}\right) \text{ et } G'\left(i\frac{x-1}{x+1}\right)$$

que je viens de définir.

Soit alors:

$$\theta(x) = e^{-G\left(i\frac{x+1}{x-1}\right) - G'\left(i\frac{x-1}{x+1}\right)} = e^{-G-G'}.$$

On voit que $\theta(x)$ est une fonction de x holomorphe dans tout le plan et n'admettant d'autres points singulièrs que deux points singulièrs essentiels $x = 1$ et $x = -1$.

Je dis maintenant que

$$\lim F(x)\theta(x) = 0$$

quand x tend vers -1 ou vers $+1$ *en suivant la circonférence* C.

En effet

$$F\theta = Fe^{-G}e^{-G'}.$$

Si x tend vers $+1$, Fe^{-G} tend vers 0 et $e^{-G'}$ vers une limite finie; si x tend vers -1, $Fe^{-G'}$ tend vers 0 et e^{-G} vers une limite finie.

Soit donc

$$x = e^{i\omega},$$

ω étant réel; x est alors sur la circonférence C; le produit $F(x)\theta(x)$ peut ainsi être regardée un instant comme une fonction de ω. Cette fonction est analytique sur tout arc de la circonférence C qui ne contient ni le point $x = +1$, ni le point $x = -1$, et quand x se rapproche indéfiniment de l'un de ces deux points elle tend vers 0. Elle est donc développable par la formule de Fourier et je puis écrire:

$$F\theta = \Sigma c_m \cos m\omega + \Sigma d_m \sin m\omega,$$

ou bien encore

$$F\theta = \Sigma a_m e^{-mi\omega} + \Sigma b_m e^{mi\omega} \qquad (a_0 = 0)$$

les coëfficients a_m et b_m étant des constantes qu'il est aisé de calculer. On a en effet:

$$2\pi a_m = \int_0^{2\pi} F\theta e^{mi\omega} d\omega$$
$$2\pi b_m = \int_0^{2\pi} F\theta e^{-mi\omega} d\omega.$$

Considérons les deux développements

$$\phi(x)\theta(x) = \Sigma a_m x^{-m}$$
$$\psi(x)\theta(x) = \Sigma b_m x^{m}.$$

Le premier de ces deux développements est convergent à l'extérieur de la circonférence C et sur la circonférence elle-même, mais diverge à l'intérieur de cette circonférence; le second développement au contraire converge à l'intérieur de C et sur la circonférence elle-même, mais diverge à l'extérieur de C.

Sur la circonférence elle-même, on a

$$\phi(x)\theta(x) + \psi(x)\theta(x) = \Sigma a_m x^{-m} + \Sigma b_m x^m = F(x)\theta(x)$$

et par conséquent:

$$\phi(x) + \psi(x) = F(x).$$

Pour reconnaître si cette égalité a encore lieu pour les valeurs de x qui n'appartiennent pas à cette circonférence, il faut chercher à prolonger analytiquement $\phi(x)$ à l'intérieur de C et $\psi(x)$ à l'extérieur de C.

Nous avons:

$$2\pi i a_m = i\int_0^{2\pi} F\theta e^{mi\omega} d\omega = \int F\theta x^{m-1} dx,$$

$$2\pi i b_m = i\int_0^{2\pi} F\theta e^{-mi\omega} d\omega = \int F\theta x^{-m-1} dx,$$

les intégrales étant étendues à la circonférence C tout entière. Si je désigne alors pour éviter toute confusion par la lettre z un point de la circonférence C et par la lettre x un point n'appartenant pas à cette circonférence, il viendra:

$$2i\pi a_m = \int F(z)\theta(z) z^{m-1} dz; \quad 2i\pi b_m = \int F(z)\theta(z) z^{-m-1} dz,$$

et par conséquent si x est extérieur à C

$$2i\pi\phi(x)\theta(x) = 2i\pi\Sigma a_m x^{-m} = \int F(z)\theta(z)^{(\underline{\Sigma z^{m-1} x^{-m}}) dz},$$

ou enfin:

$$2i\pi\phi(x)\theta(x) = \int \frac{F(z)\theta(z)dz}{x - z};$$

et si x est au contraire intérieur à C,

$$2i\pi\psi(x)\theta(x) = 2i\pi\Sigma b_m x^m = \int F(z)\theta(z)\Sigma z^{-m-1} x^m dz,$$

ou

$$2i\pi\psi(x)\theta(x) = \int \frac{F(z)\theta(z)dz}{z - x}. \qquad (10)$$

Toutes ces intégrales doivent être prises le long de C.

Ces intégrales ne définissent encore les fonctions ϕ et ψ, la première qu'à l'extérieur de C seulement et la seconde qu'à l'intérieur de C seulement.

Mais on peut modifier le contour d'intégration.

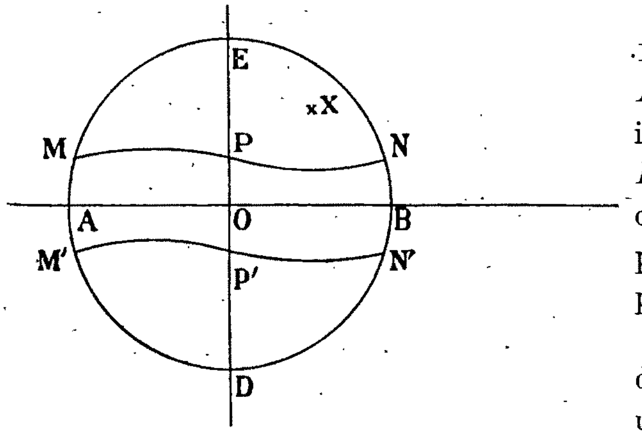


FIG. 1.

Je représente sur la figure 1 l'axe des quantités réelles AOB, l'axe des quantités imaginaires DOE, et la circonférence $BNEMAM'DN'B$ qui n'est autre chose que la circonférence C. Le point A est le point $x = -1$ et le point B est le point $x = +1$.

Joignons deux points M et N de la moitié supérieure de C par un arc de courbe quelconque MPN restant tout entier dans la moitié supérieure du cercle limité par C. Joignons de même deux points M' et N' de la moitié inférieure par un arc de courbe C.

On peut remplacer le contour d'intégration C par le contour $BNPMAM'P'N'B$ que j'appellerai C'.

Je dis que si x est extérieur à C, on aura encore

$$2i\pi\phi(x)\theta(x) = \int \frac{F(z)\theta(z)dz}{x-z},$$

l'intégrale étant prise le long de C', ou en d'autres termes que l'intégrale :

$$\int \frac{F(z)\theta(z)dz}{x-z} \tag{11}$$

prise le long de C' est égale à cette même intégrale prise le long de C.

Il suffit de montrer que cette même intégrale prise le long du contour $NEMPN$ ou le long du contour $N'P'M'DN'$ est nulle.

En effet $F(z)\theta(z)$ est holomorphe sauf sur l'axe des quantités réelles, $\frac{1}{x-z}$ est holomorphe sauf pour $z = x$. Si donc x est extérieur à C, la fonction sous le signe $\int$ sera holomorphe tant à l'intérieur du contour $NEMPN$ qu'à l'intérieur du contour $N'P'M'DN'$; ce qui démontre la proposition énoncée.

Mais l'intégrale (10) prise le long de C', définit une fonction de x qui reste holomorphe pour tous les points extérieurs à C', et comme on peut rapprocher les deux arcs MPN et $M'P'N'$ autant que l'on veut de la droite AOB, on peut définir ainsi la fonction $\phi(x)$ pour toutes les valeurs de x sauf pour le segment AOB, c'est à dire pour le segment $(-1, +1)$ qui sert de diamètre à la circonférence C.

La fonction $\phi(x)$ ainsi définie est holomorphe sauf pour les points qui appartiennent à ce segment $(-1, +1)$.

Je me propose maintenant de démontrer qu'on aura pour un point x intérieur à C

$$\phi(x) + \psi(x) = F(x).$$

Supposons en effet que le point x vienne en X, c'est à dire à l'intérieur du contour $NEMPN$. L'expression $-2i\pi\psi(x)\theta(x)$ sera égale à l'intégrale (11) prise le long de C; l'expression $2i\pi\phi(x)$ sera égale à l'intégrale (10) prise le long de C'.

Par conséquent l'expression:

$$-2i\pi\theta(\phi + \psi)$$

sera égale á l'intégrale (10) prise le long de $NEMPN$ qui est égale à $2i\pi F(x)\theta(x)$ en vertu du théorème de Cauchy, plus l'intégrale (10) prise le long de $N'P'M'DN'$ qui est nulle en vertu du même théorème; il vient donc

$$\phi + \psi = F. \qquad \text{C. Q. F. D.}$$

On définirait de la même manière la fonction $\psi(x)$ à l'extérieur de C. Il suffit pour cela de prendre l'intégrale (10) le long d'un contour C'' formé en remplaçant les arcs NEM et $M'DN'$ de la circonférence C par deux arcs de courbe quelconques NQM et $M'Q'N'$ situés en dehors de C et ne coupant pas l'axe des quantités réelles. On pourra alors choisir ces deux arcs de courbe de telle façon que le point x quel qu'il soit se trouve à l'intérieur de C''.

L'intégrale (10) prise le long de C'' est alors égale à $-2i\pi\psi(x)\theta(x)$ ce qui définit la fonction $\psi(x)$.

On verrait aussi que $\psi(x)$ est holomorphe dans tout le plan et qu'elle n'admet d'autres singularités que deux coupures qui sont les deux segments $(-\infty, -1)$ et $(+1, +\infty)$.

On démontrerait d'ailleurs par un raisonnement identique à celui qui précède que l'on a à l'extérieur de C:

$$\phi + \psi = F.$$

Cette égalité a donc lieu dans tout le plan; c'est à dire qu'on aura

$$\phi(x) + \psi(x) = f(x)$$

dans la moitié supérieure du plan et

$$\phi(x) + \psi(x) = f_1(x)$$

dans la moitié inférieure. C. Q. F. D.

Les fonctions ϕ et ψ ne sont d'ailleurs pas les seules qui jouissent de cette propriété. Si en effet $\lambda(x)$ est une fonction de x n'ayant d'autre point singulier que deux points singuliers essentiels $+1$ et -1, la fonction $\phi(x)+\lambda(x)$ n'aura d'autre singularité qu'une coupure $(-1, +1)$ et la fonction $\psi(x)-\lambda(x)$ n'aura d'autre singularité que deux coupures $(-\infty, -1)$ et $(+1, +\infty)$.

On aura d'ailleurs dans tout le plan :

$$[\phi(x)+\lambda(x)]+[\psi(x)-\lambda(x)]=F(x).$$

*On the Computation of Covariants by Transvection.**

By Emory McClintock.

Let $A = A_0x^l + A_1x^{l-1}y + A_2x^{l-2}y^2 + \dots + A_ly^l$ and $B = B_0x^m + B_1x^{m-1}y + \dots + B_my^m$ be covariants of the quantic $U = ax^n + nbx^{n-1}y + \frac{1}{2}n(n-1)cx^{n-2}y^2 + \dots$ If α_x, α_y, represent $\frac{d}{dx}$, $\frac{d}{dy}$, operating only on A, and β_x, β_y, represent $\frac{d}{dx}$, $\frac{d}{dy}$, operating only on B, then, k being any number not greater than either l or m,

$$(A, B)_k = (\alpha_x\beta_y - \beta_x\alpha_y)^k AB, \tag{1}$$

where $(A, B)_k$ represents the k^{th} transvectant of A and B, itself a third covariant, which let us denote by $C = C_0x^{l+m-2k} + C_1x^{l+m-2k-1}y + \dots$ That the effect of this operation of transvection ("ueberschiebung") is to produce a covariant is well known, as are also the theorems $XA_0 = 0$, $YA_h = (h+1)A_{h+1}$, where $X = a\frac{d}{db} + 2b\frac{d}{dc} + 3c\frac{d}{dd} + \dots$ and $Y = nb\frac{d}{da} + (n-1)c\frac{d}{db} + \dots$; as well as the converse theorem that if $X\phi = 0$, ϕ is, if homogeneous, the leading coefficient of a covariant.

The actual performance of the operation indicated in (1) will effect the computation of C in all its details. It is, however, extremely troublesome. An easier method, no doubt usually followed, is to obtain by that means the leading coefficient only, C_0, and to derive C_1, $C_2, \dots$ successively from C_0, expressed in terms of $a, b, \dots$, by repeated applications of the operation Y. For the first step I am accustomed to use the formula

$$C_0 = p^{(k}A_0B_k - qp^{(k-1}A_1B_{k-1} + q^{(2}p^{(k-2}A_2B_{k-2} + \dots \pm q^{(k}A_kB_0, \tag{2}$$

where $p = l - k + 1$, $q = m - k + 1$, $p^{(k} = p(p+1)(p+2)\dots(p+k-1)$. This will be found, on examination, to embody, omitting a common numerical factor, the results of the operation indicated in (1) for all terms independent of y. Something equivalent to (2) must no doubt be employed for the evaluation of C_0 by every one who has to do with the computation of covariants by transvection. If we undertake to frame, in terms of $A_0, A_1, \ldots, B_0, \ldots$, corresponding formulæ for C_1, C_2, etc., we find them (except perhaps, as we shall see, when $k = 1$) becoming too complex for use, and it is easier, as just stated, to derive the other coefficients directly from C_0 by applying repeatedly the operation Y.

---

* Read before the New York Mathematical Society, Jan. 2, 1892.

What cannot be done with formulæ, however, towards assigning the numerical coefficients resulting from (1), can be accomplished by a tabular construction, by the aid of which I find it easy to compute C_1, C_2, etc., by means of $A_0, A_1, \ldots, B_0, \ldots$, without performing the operation Y. It is perhaps better to present this method at first by a numerical example. Let it be required to calculate the third transvectant of two covariants of the fourth and sixth orders respectively. (This is the process recommended, for instance, by Clebsch and Gordan for producing degree 6, order 4, of the quintic.) Let us call the quartic A, and the sextic B (the choice is immaterial), and let us form a table having as arguments at the side the coefficients of A and at the top the coefficients of B, the spaces of the table to be occupied by the numerical multipliers, respectively, of the product of the argument at the side by the argument at the top, in the resulting expression for the transvectant desired, say C. We know that the latter can contain no expression $A_g B_h$ in which $g + h < k$. All spaces for which the sum of the subscripts of the arguments is less than k, that is, less than 3 in this case, are therefore filled with zeros. We know that $g + h = k$ for all terms composing C_0, that $g + h = k + 1$ for all composing C_1, and so on. Therefore the diagonal $A_0B_3 \ldots A_3B_0$ will contain the multipliers for the terms of C_0, the next diagonal for those of C_1, and so on. We fill the diagonal A_0B_3, A_1B_2, etc., by means of formula (2), which in this case (where $l = 4$, $m = 6$, $k = 3$, $p = 2$, $q = 4$),

	B_0	B_1	B_2	B_3	B_4	B_5	B_6
A_0	0	0	0	3	12	30	60
A_1	0	0	—3	—6	—6	0	15
A_2	0	5	4	0	—4	—5	0
A_3	—15	0	6	6	3	0	0
A_4	—60	—30	—12	—3	0	0	0

dropping the common factor 8, gives $C_0 = 3A_0B_3 - 3A_1B_2 + 5A_2B_1 - 15A_3B_0$. These figures being inserted in the table, the remaining spaces in the line A_0, at top, are filled successively by multiplying the number to the left by the subscript of B for the space to be filled, and dividing the result in the first instance by 1, in the second by 2, and so on. The remaining spaces in the column B_0 are completed in like manner, by multiplying the number above by the subscript of A, the rule for division being the same. The figures -6, 4, 0, completing the diagonal of C_1, are now obtained by performing both operations, and dividing the sum by 1; those completing the diagonal of C_2 are next computed likewise, dividing by 2, and so on. Thus, for A_1B_3 in C_1, the number above (3) is multiplied by the subscript of A(1), and the number to the left (-3) by the subscript of B(3), the results (3 and -9) added together and divided by 1 giving -6. Again, for A_3B_4 in C_4, the number above (-4) is multiplied by 3, and that to the left (6) by 4, producing $(-12 + 24) \div 4 = 3$. In all cases, the process to be pursued is embodied in the formula

$$x = (s\sigma + t\tau) \div r, \tag{3}$$

where x is the number required for the space A_sB_t, σ that appearing in the space above, τ that to the left, and r the number of the diagonal (counting the first as 0), or subscript of C. It is obvious that $r = s + t - k$.

In proof of (3), we observe that C_{r-1} consists of products having subscripts amounting in each case to $s + t - 1$, and including $\sigma A_{s-1}B_t + \tau A_sB_{t-1}$. The operation Y, which changes A_{g-1} into gA_g, will, if performed on those products, produce a succession of products having subscripts of the joint value of $s + t$, including among them the form A_sB_t. Since we have $YA_{s-1} = sA_s$, $YB_{t-1} = tB_t$, $YC_{r-1} = rC_r$, it follows that the coefficient of A_sB_t in YC_{r-1} is on the one hand equal to rx, and on the other hand equal to $s\sigma + t\tau$, whence (3) follows immediately. Since $XC_0 = 0$, when C_0 is defined as in (2), showing that C_0 is a new covariant, the original definition of transvection contained in (1) is not necessary, when the only object in view is the production of new covariants.

If we call the evaluation of (1) the first method of calculation, and the repeated use of Y upon the full expression of C_0 in terms of $a, b, \ldots$ the second method, we may explain this third method as equivalent to the second, except that the actual performance of Y is avoided, while the work is assisted by recourse to the tabular form and to formula (3).

When $k = 1$ (the case of the Jacobian), the table takes a simple form, in that all rows and columns must be in arithmetical progression. Thus for $l = 4$, $m = 6$, $k = 1$:

	B_0	B_1	B_2	B_3	B_4	B_5	B_6
A_0	0	2	4	6	8	10	12
A_1	−3	−1	1	3	5	7	9
A_2	−6	−4	−2	0	2	4	6
A_3	−9	−7	−5	−3	−1	1	3
A_4	−12	−10	−8	−6	−4	−2	0

That this must be the case follows from (1), viz.,

$$(\mathrm{A}, \mathrm{B}) = \frac{dA}{dx}\frac{dB}{dy} - \frac{dB}{dx}\frac{dA}{dy}.$$

Here the coefficient of A_0B_g is lg; that of A_1B_g is $(l-1)g-(m-g) = lg - m$; that of A_2B_g is $(l-2)g - 2(m-g) = lg - 2m$; and so on, with m as a constant difference. When $k = 1$, therefore, we are able to lay down a sufficiently simple formula,

$$x = (l - s)t - s(m - t) = lt - ms. \tag{4}$$

This is so obvious, and the evaluation of Jacobians has been performed so often as compared with other transvectants, that it must have been observed repeatedly. The use of (4), however, is less advantageous than the tabular method with constant differences.

The operation (1) produces, as we have just seen in (4), a function of the subscripts of the first degree when $k = 1$. Similarly, when $k = 2$, it produces a function of the second degree, and so on. The rows and columns, therefore, when $k = 1$, have constant first differences; when $k = 2$, constant second differences, and so on. While this observation is chiefly serviceable when $k = 1$, it will be found useful for other small values of k, particularly in complicated cases. For example, if $l = 4$, $m = 6$, $k = 2$, the constant second difference of each row is 4, and of each column 10. From the method of forming the 0 row and 0 column it follows in all cases that the constant (k^{th}) difference of each row is the first number in the 0 row, and that of each column the first number in the 0 column.

Since by interchanging x and y the whole process might be performed backwards, calling A_l A_0, B_m B_0, etc., the numbers forming our table are always the same at opposite points, except that the sign may differ; that is to say, the multiplier for A_sB_l is numerically the same as for $A_{l-s}B_{m-l}$. This property, together with the constant k^{th} differences, makes the computation of the table a matter of little difficulty in most cases. It is not, indeed, really necessary to carry the table more than half way. When the evaluation of the resulting covariant has been effected as far as it can be by the help of half the table, the remaining complementary terms can be written off, as usual, by substituting — in the case of the sextic, for example — g for a, f for b, and so on. Yet it is safer to carry the work out completely, and to check the results by verifying the complements.

When one of the covariants employed is the quantic itself, we may write a at the top of the first column, nb (say, for the sextic, $6b$) at the top of the second, and so on, in lieu of B_0, B_1, ..., or in addition to those headings. A better way, however, will be explained shortly. We may, of course, in simple cases, rearrange the whole table to suit our convenience, even substituting diagonals for columns, and *vice versa*. For example, I have used this table to compute the fourth transvectant of the covariant of the sextic of degree 2, order 4, with the sextic itself:

	For C_0.	For C_1.	For C_2.
$ae - 4bd + 3c^2$	e	$2f$	g
$2af - 6be + 4cd$	$-d$	$-2e$	$-f$
$ag - 9ce + 8d^2$	c	$2d$	e
$2bg - 6cf + 4de$	$-b$	$-2c$	$-d$
$cg - 4df + 3e^2$	a	$2b$	c

Performing these multiplications, and dividing throughout by 2, we have

$$\begin{aligned} C_0 &= acg - 3adf + 2ae^2 - b^2g + 3bcf - bde - 3c^2e + 2cd^2, \\ C_1 &= -bcg - 8bdf + 9be^2 + 9c^2f - 17cde + adg + 8d^3 - aef, \\ C_2 &= aeg - 3bdg + 2c^2g - af^2 + 3bef - cdf - 3ce^2 + 2d^2e. \end{aligned}$$

We have here, as will be seen, the coefficients of the quadratic covariant of the third degree. This sort of table may be used to advantage whenever B = quantic and k = order of A.

I have, in the current number (January, 1892) of the *Bulletin of the New York Mathematical Society*, mentioned facts which have led me to believe that the simplest expression for any desired groundform, whose source is not that of a groundform of the quantic next lower, is that which is produced, when the operation is possible, by simple transvection from the nearest available covariant; using the phrase "simple transvection" as meaning "transvection with the quantic itself."

It will be found that for simple transvection the multipliers can be considerably reduced by incorporating with them the numbers at the head of the columns (coefficients of b, c, ...) and striking out common factors. The labor of doing so may, however, be obviated by applying formulæ different from, but derived from (2) and (3). Let s, as before, denote the row, and t the column, t being in this case the weight of that literal coefficient of the quantic which appears at the head of the column. On occasion, we may write a_0 for a, a_1 for b, and so on. In this case $\mathrm{B}=\mathrm{U}$, $m=n$; $q=n-k+1$, $\mathrm{B}_0=a$, $\mathrm{B}_1=nb$, $\mathrm{B}_2=\frac{1}{2}n(n-1)c$, and so on. If now we reverse the order of the terms in (2), dividing throughout by $\pm q^{(k)}$, we have for this case

$$\mathrm{C}_0=\mathrm{A}_k a-p\mathrm{A}_{k-1}b+\frac{1}{2!}p^{(2)}\mathrm{A}_{k-2}c-\dots \tag{5}$$

Again, if we assume all other numbers in the table to be similarly divided, we have the right to employ (3) for the determination of the other numbers, and to modify it similarly by appropriate substitutions. In this way we obtain

$$z=(s\zeta+u\theta)\div r, \tag{6}$$

where z is the multiplier desired for A_s, column t, ζ the number above, θ that to the left, r the same as before, and $u=n-t+1$. It is to be observed that z and ζ are respectively $n(n-1)\dots(n-t+1)/t!$ times larger than s and σ, and that θ is $n(n-1)\dots(n-t)/t-1!$ times larger than r. We may say, therefore, that (6) is derived from (3) by making these substitutions and multiplying throughout by $n(n-1)\dots(n-t+1)/t!$, which leaves θ with a coefficient $t(n-t+1)/t=u$. Or, we may obtain (6) *de novo* by remarking that $Y\mathrm{A}_{s-1}=s\mathrm{A}_s$, $Ya_{t-1}=(n-t+1)a^t$, $Y\mathrm{C}_{r-1}=r\mathrm{C}_r$, so that the coefficient of $\mathrm{A}_s a_t$ in $Y\mathrm{C}_{r-1}$ is on the one hand rz, on the other hand $s\zeta+u\theta$.

The formation of a table for simple transvection may be illustrated by the case $n=6$, $l=4$, $k=2$, $p=3$:

	0	1	2	3	4	5	6
	a	b	c	d	e	f	g
A_0	0	0	6	24	36	24	6
A_1	0	—3	—9	— 6	6	9	3
A_2	1	0	—9	—16	—9	0	1
A_3	3	9	6	— 6	—9	—3	0
A_4	6	24	36	24	6	0	0

The upper row, beginning with column k, consists of the binomial coefficients of the degree $n-k$, each multiplied by $p^{(k}$; the zero column is formed by writing 1 opposite A_k, then $k+1$, $\frac{1}{2}(k+1)^{(2}$, $\frac{1}{3!}(k+1)^{(3}$, and so on; and the first diagonal, beginning with the space $A_k a_0$, consists of 1, p, $\frac{1}{2}p^{(2}$, and so on. The columns (alone) can be completed or verified by constant k^{th} differences. For another illustration, let us take a Jacobian, $n=6$, $l=4$, $k=1$, $p=4$:

	0	1	2	3	4	5	6
	a	b	c	d	e	f	g
A_0	0	—4	—20	—40	—40	—20	—4
A_1	1	2	— 5	—20	—25	—14	—3
A_2	2	8	10	0	—10	— 8	—2
A_3	3	14	25	20	5	— 2	—1
A_4	4	20	40	40	20	4	0

That case of simple transvection in which $k=l$, so that $p=1$, supplies a numerical table which may be written down without calculation, and the table may be simplified further by substituting columns for diagonals. (An instance has already been shown.) Under the general rule, when $n=6$, $l=4$, $k=4$, $p=1$, we should put the table in this form:

	0	1	2	3	4	5	6
	a	b	c	d	e	f	g
A_0	0	0	0	0	1	2	1
A_1	0	0	0	—1	—2	—1	0
A_2	0	0	1	2	1	0	0
A_3	0	—1	—2	—1	0	0	0
A_4	1	2	1	0	0	0	0

The simplification now in question consists in reducing the number of columns to $n-l+1$, one column for each coefficient of C in the result; in writing the binomial coefficients for the degree $n-l$ at the head of the columns; in affecting $A_1, A_3, \ldots$, having odd subscripts, with the negative sign; in writing a at the foot of the first column, b above it, and so on, the rows being then completed in alphabetical order. The illustrative table just given then takes this shape:

	For c_0	For c_1	For c_2
	$\times 1$	$\times 2$	$\times 1$
A_0	e	f	g
$-A_1$	d	e	f
A_2	c	d	e
$-A_3$	b	c	d
A_4	a	b	c

The coefficient of c_0 when $p=1$ is, by (5), $A_k a - A_{k-1} b + A_{k-2} c - \ldots$, the numerical multipliers being equal, with alternate signs. But by (6), when the numerical multipliers, say for c_{r-1}, are equal with alternate signs, we have $\theta = -\zeta$, $rz = \theta(u-s) = \theta(n-t-s+1) = \theta(n-l+1-r)$. If $n-l$, which is the order of C, be represented by λ, we find thus that the numerical value of each multiplier for c_0 is 1, for c_1 is λ, for c_2 is $\frac{1}{2}\lambda(\lambda-1)$, and so on according to the binomial series, the multipliers of each column having alternate, and of each row the same, signs, as constructed in the foregoing table.

Transformation of a System of Independent Variables.

By J. C. Fields.

Let $x, y, z, \ldots.$ be any number of independent variables; $u, v, w, \ldots.$ the same number of independent functions of $x, y, z, \ldots.$ and ϕ any function of $u, v, w, \ldots.$; being therefore also a function of $x, y, z, \ldots.$ We wish to transform any differentiation of ϕ with regard to $x, y, z, \ldots.$ into differentiations of ϕ with regard to $u, v, w, \ldots.$ or in other words to obtain for any differential coefficient $\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s\left(\frac{d}{dz}\right)^t \ldots.\phi$ its equivalent expression of the form $\sum A_{\lambda,\mu,\nu\ldots.}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu\left(\frac{d}{dw}\right)^\nu\ldots.\phi$, where $A_{\lambda,\mu,\nu\ldots.}$ is a function of the differential coefficients of $u, v, w, \ldots.$ with respect to $x, y, z, \ldots.$ and independent of the form of ϕ. First supposing ϕ and u to be functions of but one independent variable x, we will have $\left(\frac{d}{dx}\right)^r\phi = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda\phi$, an equivalence which may also be stated in the form $\left(\frac{d}{dx}\right)^r = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda$, where A_λ is a function of differential coefficients of u with regard to x and independent of the form of ϕ. Assume that, α being an arbitrary parameter, we have in any manner obtained an expression for the r^{th} derivative of $\rho^{\alpha u}$ in the form $\left(\frac{d}{dx}\right)^r\rho^{\alpha u} = \sum B_\rho\alpha^\rho.\rho^{\alpha u}$, where B_ρ is independent of α. Since $\rho^{\alpha u}$ is a particular form of $\phi(u)$, we will have also evidently $\left(\frac{d}{dx}\right)^\rho\rho^{\alpha u} = \sum A_\lambda\left(\frac{d}{du}\right)^\lambda\rho^{\alpha u} = \sum A_\lambda\alpha^\lambda.\rho^{\alpha u}$, where A_λ is the same as in the general case, and being independent of the form of ϕ is in this case independent of α. The two expressions for the r^{th} derivative of $\rho^{\alpha u}$ must be equal and therefore $\sum A_\lambda\alpha^\lambda.\rho^{\alpha u} = \sum B_\rho\alpha^\rho.\rho^{\alpha u}$. Consequently

$\sum A_\lambda \alpha^\lambda = \sum B_\rho \alpha^\rho$, and since this equation holds for any value of α while also the coefficients A and B do not involve α, we must have $A_\lambda \equiv B_\lambda$. We have then $\left(\frac{d}{dx}\right)^r = \sum A_\lambda \left(\frac{d}{du}\right)^\lambda = \sum B_\lambda \left(\frac{d}{du}\right)^\lambda$. In order now to find the general formula for the transformation of the r^{th} derivative of ϕ, we first find the r^{th} derivative with regard to x of $\rho^{\alpha u}$ in the form $\sum B_\lambda \alpha^\lambda . \rho^{\alpha u}$, and substituting $\frac{d}{du}$ for α in the summation of $\sum B_\lambda \alpha^\lambda$ obtain thence a formula which holds in general—

$$\left(\frac{d}{dx}\right)^r = \sum B_\lambda \left(\frac{d}{du}\right)^\lambda. \tag{1}$$

We proceed to find the r^{th} derivative of $\rho^{\alpha u}$. The coefficient of ξ^r in the expression of a function $F(x+\xi)$ in powers of ξ is $\frac{1}{r!}\left(\frac{d}{dx}\right)^r F(x)$; consequently $\frac{1}{r!}\left(\frac{d}{dx}\right)^r \rho^{\alpha u}$ is equal to the coefficient of ξ^r in the expansion in powers of ξ of $\rho^{\alpha\left(u + u'\xi + \frac{u''}{2!}\xi^2 \ldots\right)}$

$$= \rho^{\alpha u} . \rho^{\alpha u'\xi} . \rho^{\alpha \frac{u''}{2!}\xi^2} \ldots \rho^{\alpha \frac{u^{(\kappa)}}{\kappa!}\xi^\kappa} \ldots = \rho^{\alpha u} \sum \frac{(u'\xi)^{p_1} \left(\frac{u''}{2!}\xi^2\right)^{p_2} \ldots \left(\frac{u^{(\kappa)}}{\kappa!}\xi^\kappa\right)^{p_\kappa} \ldots}{p_1!\, p_2! \ldots p_\kappa \ldots} \alpha^{p_1 + p_2 + \ldots}$$

We have therefore

$$\frac{1}{r!}\left(\frac{d}{dx}\right)^r \rho^{\alpha u} = \rho^{\alpha u} \sum \frac{(u')^{p_1} \left(\frac{u''}{2!}\right)^{p_2} \ldots \left(\frac{u^{(\kappa)}}{\kappa!}\right)^{p_\kappa} \ldots}{p_1!\, p_2! \ldots p_\kappa! \ldots} \alpha^{p_1 + p_2 + \ldots} \equiv \frac{1}{r!} \rho^{\alpha u} \sum B_\lambda \alpha^\lambda \tag{2}$$

where the summation includes all terms such that $p_1 + 2p_2 + \ldots + \kappa p_\kappa + \ldots = r$.

Replacing α by $\frac{d}{du}$ we hence obtain from formula (1),

$$\frac{1}{r!}\left(\frac{d}{dx}\right)^r = \sum \frac{(u')^{p_1} (u'')^{p_2} \ldots (u^{(\kappa)})^{p_\kappa} \ldots}{p_1!\, p_2!\, (2!)^{p_2} \ldots p_\kappa!\, (\kappa!)^{p_\kappa} \ldots} \left(\frac{d}{du}\right)^\lambda \tag{3}$$

where $\lambda = p_1 + p_2 + \ldots p_\kappa + \ldots$, the summation being subject as before to the conditions $p_1 + 2p_2 + \ldots + \kappa p_\kappa + \ldots = r$. This theorem was first obtained by Faà de Bruno,[*] who proves it by induction. Proofs have also been given by M. Bertrand[†] and by M. de Presle.[‡] Formula (3) may be neatly expressed in the form of a symbolic determinant. On differentiating $(r-1)$ times in succession the differential equation $\frac{dy}{dx}-\frac{dz}{dx}y=0$ of which a solution is $y=\rho^z$, and solving for $\frac{d^ry}{dx^r}$, we obtain in the form of a determinant an expression for the r^{th} derivative of ρ^z, which may be written

---

[*] Annales de Tortolini, 1855; reproduced in Théorie de Formes Binaires, p. 804.

[†] Cal. Diff., p. 808.

[‡] Bulletin de la Société Mathématique, Vol. XVI, p. 157.

$$\left(\frac{d}{dx}\right)^r\rho^z=\begin{vmatrix} z' , & \frac{z''}{1!}, & \frac{z'''}{2!}, \dots & \frac{z^{(r)}}{(r-1)!} \\ -(r-1), & z', & \frac{z''}{1!}, \dots & \frac{z^{(r-1)}}{(r-2)!} \\ 0 , & -(r-2), & z', \dots & \frac{z^{(r-2)}}{(r-3)!} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \;\dots & -1, z' \end{vmatrix}\rho^z$$

In this expression substituting au for z and afterwards $\frac{d}{du}$ for a, we will have by (1),

$$\left(\frac{d}{dx}\right)^r=\begin{vmatrix} u'\frac{d}{du}, & \frac{u''}{1!}\frac{d}{du}, & \frac{u'''}{2!}\frac{d}{du}\dots & \frac{u^{(r)}}{(r-1)!}\frac{d}{du} \\ -(r-1), & u'\frac{d}{du}, & \frac{u''}{1!}\frac{d}{du}\dots & \frac{u^{(r-1)}}{(r-2)!}\frac{d}{du} \\ 0 , & -(r-2), & u'\frac{d}{du}\dots & \frac{u^{(r-2)}}{(r-3)!}\frac{d}{du} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \;\dots & -1, u'\frac{d}{du} \end{vmatrix} = \tag{4}$$

$$\begin{vmatrix} u' , & \frac{u''}{1!}, & \frac{u'''}{2!}\dots & \frac{u^{(r)}}{(r-1)!} \\ -(r-1)\left(\frac{d}{du}\right)^{-1}, & u' , & \frac{u''}{1!}\dots & \frac{u^{(r-1)}}{(r-2)!} \\ 0 , & -(r-2)\left(\frac{d}{du}\right)^{-1}, & u' \;\dots & \frac{u^{(r-2)}}{(r-3)!}\left(\frac{d}{du}\right)^r \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \;\dots & -\left(\frac{d}{du}\right)^{-1}, u' \end{vmatrix}$$

where of course $\frac{d}{du}$ is supposed not to operate upon any element of the determinant itself, a supposition which in the case of the second form of determinant may be taken account of by operating first in any term with those elements which involve $\left(\frac{d}{du}\right)^{-1}$.

We might also notice a generalization of the well-known theorem $\left(\frac{d}{du}\right)^r \rho^{\alpha u} v = \rho^{\alpha u}\left(\frac{d}{du}+\alpha\right)^r v$. For brevity writing $\Delta_r\left(\frac{d}{du}\right)$ instead of the second determinant of (4) we have—

$$\left(\frac{d}{dx}\right)^r \rho^{\alpha u} v = \Delta_r\left(\frac{d}{du}\right) \cdot \left(\frac{d}{du}\right)^r \cdot \rho^{\alpha u} v = \rho^{\alpha u} \Delta_r\left(\frac{d}{du}+\alpha\right) \cdot \left(\frac{d}{du}+\alpha\right)^r \cdot v =$$

$$\rho^{\alpha u}\left|\begin{array}{cccc} u', & \frac{u''}{1!}, & \frac{u'''}{2!} \cdots\cdots & \frac{u^{(r)}}{(r-1)!} \\ -(r-1)\left(\frac{d}{du}+\alpha\right)^{-1}, & u', & \frac{u''}{1!} \cdots\cdots & \frac{u^{(r-1)}}{(r-2)!} \\ 0 -(r-2)\left(\frac{d}{du}+\alpha\right)^{-1}, & u', & \cdots\cdots & \frac{u^{(r-2)}}{(r-3)!} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0 & 0 \cdots\cdots & -\left(\frac{d}{du}+\alpha\right)^{-1}, u' \end{array}\right| \left(\frac{d}{du}+\alpha\right)^r \cdot v \quad (5)$$

Our method of procedure in dealing with a function of several variables will be the same as that employed in the case of one variable. If ϕ be a function of any number of variables $u, v, w, \ldots\ldots$ each of which is a function of the same number of independent variables $x, y, z, \ldots\ldots$ we will have $\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s \ldots\ldots \phi =$ $\sum A_{\lambda,\mu,\ldots\ldots}\left(\frac{d}{du}\right)^\lambda\left(\frac{d}{dv}\right)^\mu \ldots\ldots \phi$, an expression of finite degree in $\frac{d}{du}, \frac{d}{dv}, \ldots\ldots$ the coefficients A of which are functions of the differential coefficients of $u, v, \ldots\ldots$ with regard to $x, y \ldots\ldots$, and independent of the form of ϕ. Suppose we have in any manner obtained a formula

$$\left(\frac{d}{dx}\right)^r\left(\frac{d}{dy}\right)^s \ldots\ldots \rho^{\alpha u+\beta v+\ldots\ldots} = \sum B_{\rho, m, \ldots\ldots}\alpha^\rho\beta^m \ldots\ldots \rho^{\alpha u+\beta v+\ldots\ldots}$$

where the coefficients B are independent of the arbitrary parameters $\alpha, \beta, \ldots.$ Since $\rho^{\alpha u + \beta v + \cdots}$ is a particular form of ϕ, we will have also

$$\left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots \rho^{\alpha u + \beta v + \cdots} = \sum A_{\lambda, \mu, \ldots} \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots \rho^{\alpha u + \beta v + \cdots} =$$

$\sum A_{\lambda, \mu, \ldots} \alpha^\lambda \beta^\mu \ldots \rho^{\alpha u + \beta v + \cdots}$ where the coefficients A are independent of the parameters $\alpha, \beta, \ldots.$ Consequently

$$\sum A_{\lambda, \mu} \ldots \alpha^\lambda \beta^\mu \ldots = \sum B_{\rho, \pi, \ldots} \alpha^\rho \beta^\pi \ldots$$

and the functional coefficients A and B not involving the arbitrary parameters, we will have $A_{\lambda, \mu \ldots} = B_{\lambda, \mu \ldots}$, and hence

$$\sum A_{\lambda, \mu, \ldots} \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots = \sum B_{\lambda, \mu, \ldots} \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots$$

Our process therefore will be to first obtain the formula

$$\left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots \rho^{\alpha u + \beta v \cdots} = \sum B_{\lambda, \mu \ldots} \alpha^\lambda \beta^\mu \ldots \rho^{\alpha u + \beta v + \cdots}$$

then substitute $\frac{d}{du}, \frac{d}{dv} \ldots,$ for $\alpha, \beta, \ldots$ respectively in the summation $\sum B_{\lambda, \mu} \ldots \alpha^\lambda \beta^\mu \ldots$ obtaining thence the formula

$$\left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots = \sum B_{\lambda, \mu} \ldots \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots \tag{6}$$

which, as we have seen, must hold good in general.

We will now determine the form of the coefficients B in (6).

We know that $\frac{1}{r!}\left(\frac{d}{dx}\right)^r \frac{1}{s!}\left(\frac{d}{dy}\right)^s \ldots F(x, y, \ldots)$ is the coefficient of $\xi^r \eta^s \ldots$ in the expansion of $F(x + \xi, y + \eta, \ldots)$ in powers of $\xi, \eta, \ldots.$ If then ρ^ω is a function of $x, y, \ldots$ we will have $\frac{1}{r!\, s! \ldots} \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots \rho^\omega$ for the coefficient of $\xi^r \eta^s \ldots$ in the expansion of $\rho^{\omega + \Delta\omega + \frac{\Delta^2}{2!}\omega + \cdots} = \rho^{\rho^\Delta \omega}$ where $\Delta \equiv \xi \frac{d}{dx} + \eta \frac{d}{dy} + \ldots.$ Putting $\omega \equiv \alpha u + \beta v + \ldots$, we have

$$\frac{1}{r!\, s! \ldots} \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots \rho^{\alpha u + \beta v + \cdots} = \text{coefficient of } \xi^r \eta^s \ldots \text{ in } \rho^{\rho^{\Delta}(\alpha u + \beta v + \ldots)} \tag{7}$$

Now $\rho^{\rho^{\Delta}(\alpha u+\beta v+\ \ldots)} = \rho^{\rho^{\Delta}\alpha u} . \rho^{\rho^{\Delta}\beta v} \ldots . = \rho^{\left(\rho^{\xi\frac{d}{dx}},\ \rho^{\eta\frac{d}{dy}}\ldots\right)\alpha u} . \rho^{\rho\Delta^{\beta v}} \ldots .$

$$= \rho^{\Sigma\frac{\left(\xi\frac{d}{dx}\right)^{a}}{a!} . \Sigma\frac{\left(\eta\frac{d}{dy}\right)^{b}}{b!} \ldots . \alpha u} . \rho^{\rho^{\Delta}\beta v} \ldots . = \rho^{\Sigma\frac{\left(\xi\frac{d}{dx}\right)^{a}\left(\eta\frac{d}{dy}\right) \ldots .}{a!\ \ b!\ \ \ldots .}\alpha u} . \rho^{\rho^{\Delta}\beta v} \ldots .$$

$$= \rho^{\alpha u} . \rho^{\frac{\left(\xi\frac{d}{dx}\right)^{a_1}\left(\eta\frac{d}{dy}\right)^{b_1}\ldots .}{a_1!\ \ b_2!\ \ \ldots .}\alpha u} . \rho^{\frac{\left(\xi\frac{d}{dx}\right)^{a_2}\left(\eta\frac{d}{dy}\right)^{b_2}\ldots .}{a_2!\ \ b_2!\ \ \ldots}\alpha u} \ldots . \rho^{\rho^{\Delta}\beta v} \ldots .$$

(where in none of the exponents of ρ are all the numerical exponents $a, b, \ldots .$ zero at the same time)

$$= \rho^{\alpha u} \sum \frac{1}{p_1!\ p_2! \ldots .} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_1}\left(\eta\frac{d}{dy}\right)^{b_1}\ldots .\ \alpha u}{a_1!\ \ b_1!\ \ \ldots .}\right)^{p_1} . \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_2}\left(\eta\frac{d}{dy}\right)^{2}\ldots .\ \alpha u}{a_2!\ \ b_2!\ \ \ldots .}\right)^{p_2} \ldots . \rho^{\rho^{\Delta}\beta v} \ldots .$$

Developing $\rho^{\rho^{\Delta}\beta v}$, $\rho^{\rho^{\Delta}\gamma w} \ldots .$ in the same way, we find

$$\rho^{\rho^{\Delta}\omega} = \rho^{\omega} \sum \frac{1}{\pi(p!)\ \pi(p'!)\ \pi(p''!) \ldots .} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_1}\left(\eta\frac{d}{dy}\right)^{b_1}\ldots .\ \alpha u}{a_1!\ \ b_1!\ \ \ldots .}\right)^{p_1} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a_2}\left(\eta\frac{d}{dy}\right)^{b_2}\ldots .\ \alpha u}{a_2!\ \ b_2!\ \ \ldots .}\right)^{p_2} \ldots .$$
$$\left(\frac{\left(\xi\frac{d}{dx}\right)^{a'_1}\left(\eta\frac{d}{dy}\right)^{b'_1}\ldots .\ \beta v}{a'_1!\ \ b'_1!\ \ \ldots .}\right)^{p'_1} \left(\frac{\left(\xi\frac{d}{dx}\right)^{a'_2}\left(\eta\frac{d}{dy}\right)^{b'_2}\ldots .\ \beta v}{a'_2!\ \ b'_2!\ \ \ldots .}\right)^{p'_2} \ldots . \tag{8}$$

where for brevity the product $p_1!\ p_2! \ldots .$ is represented by $\pi(p!) \ldots .$

We will now have the coefficient of $\xi^r\eta^s \ldots .$ in $\rho^{\rho^{\Delta}\omega}$ equal to

$$\frac{1}{r!\ s! \ldots .}\left(\frac{d}{dx}\right)^{r}\left(\frac{d}{dy}\right)^{s} \ldots . \rho^{\alpha u+\beta v+\ldots .} = \rho^{\omega} \sum \frac{1}{\pi(p!)\ \pi(p'!) \ldots .}$$
$$\left(\frac{\left(\frac{d}{dx}\right)^{a_1}\left(\frac{d}{dy}\right)^{b_1}\ldots .\ u}{a_1!\ \ b_1!\ \ \ldots .}\right)^{p_1}\left(\frac{\left(\frac{d}{dx}\right)^{a_2}\left(\frac{d}{dy}\right)^{b}\ \ldots .\ u}{a_2!\ \ b_2!\ \ \ldots .}\right)^{p_2} \ldots . \left(\frac{\left(\frac{d}{dx}\right)^{a'_1}\left(\frac{d}{dy}\right)^{b'_1}\ldots .\ v}{a'_1!\ \ b'_1!\ \ \ldots .}\right)^{p'_1}\left(\frac{\left(\frac{d}{dx}\right)^{a'_2}\left(\frac{d}{dy}\right)^{b'_2}\ldots .\ v}{a'_2!\ \ b'_2!\ \ \ldots .}\right)^{p'_2} \ldots \alpha^{\Sigma p} . \beta^{\Sigma p'} \ldots \tag{9}$$

subject to the conditions $\Sigma pa + \Sigma p'a' + \ldots . = r$, $\Sigma pb + \Sigma p'b' + \ldots . = s$, etc.

We may write (9) in the form

$$\frac{1}{r!\ s! \ldots .}\left(\frac{d}{dx}\right)^{r}\left(\frac{d}{dy}\right)^{s} \ldots . \rho^{\alpha u+\beta v+\ldots .}$$
$$= \sum C\pi(\nabla u)^{p} . \pi(\nabla v)^{p'} \ldots . \alpha^{\lambda}\beta^{\mu} \ldots . \rho^{\alpha u+\beta v+\ldots .} \tag{10}$$

where by ∇u is meant any differential coefficient of u with regard to $x, y, \ldots .$ and where $\pi(\nabla u)^{p}$ designates any product of powers of differential coefficient of u, $C \ldots .$ being a constant coefficient for any term and equal in value to the reciprocal of the product of the factorials of all the exponents which appear in the term, each exponent being taken as often as it appears in the term, *e. g.* in the term which appears explicitly in (9), a_1 enters as an exponent p_1 times while p_1 occurs once, and consequently in the denominator of the coefficient

$a_1!$ appears p_1 times and $p_1!$ once. The only limitations upon the summation in (10), as we see from (9), are that the dimensions of each term in $\left(\frac{d}{dx}\right), \left(\frac{d}{dy}\right), \ldots$ must respectively equal $r, s, \ldots$ while $\lambda, \mu, \ldots$ must indicate its dimensions in $u, v, \ldots$ respectively. From (10) by (6) we now derive

$$\frac{1}{r!\, s! \ldots} \left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots = \sum C\pi(\nabla u)^p \cdot \pi(\nabla v)^{p'} \ldots \left(\frac{d}{du}\right)^\lambda \left(\frac{d}{dv}\right)^\mu \ldots \qquad (11)$$

where to repeat ourself the only conditions upon the summation are:

I. The dimensions of every term in $\frac{d}{dx}, \frac{d}{dy}, \ldots$ respectively must be $r, s, \ldots$

II. The dimensions of every term in $u, v, \ldots$ respectively must be zero.

III. The coefficient in any term is the reciprocal of the product of the factorials of all exponents appearing in the term, each being taken as often as it occurs.

That all the terms which occur in the transformed expression for $\left(\frac{d}{dx}\right)^r \left(\frac{d}{dy}\right)^s \ldots$ must satisfy the first two conditions is from the beginning self-evident, but that all the products which satisfy these conditions do occur is not so evident, nor is it immediately apparent what the values of the numerical coefficients will be.

The Deduction of Final Formulas for the Algebraic Solution of the Quartic Equation.

By Mansfield Merriman.

I.—The solution of a quartic equation depends upon that of a cubic resolvent. Let this cubic be of the form

$$y^3 + 3py^2 + 3qy + r = 0, \tag{1}$$

the three roots of which are expressed by

$$\begin{aligned} y_1 &= -p + s + t, \\ y_2 &= -p + es + e^2t, \\ y_3 &= -p + e^2s + et, \end{aligned}$$

in which e is an imaginary cube root of unity. To find s and t, let h and k first be determined from the given coefficients by

$$h = \frac{1}{2}(-2p^3 + 3pq - r), \quad k = (p^2 - q)^3, \tag{2}$$

and then

$$s = (h + \sqrt{h^2 - k})^{\frac{1}{3}}, \qquad t = (h - \sqrt{h^2 - k})^{\frac{1}{3}}. \tag{3}$$

These formulas are well known; if $p = 0$, that for y_1 becomes the formula of Cardan.

When the coefficients in (1) have such values that $h^2 - k$ is a positive quantity the cubic has one real and two imaginary roots, the values of s and t are real, and the real root is given by y_1. The imaginary roots will then be expressed in the simplest practical form by inserting the values of e and e^2. Thus, the roots are,

$$\left.\begin{aligned} y_1 &= -p + (s + t), \\ y_2 &= -p - \frac{1}{2}(s + t) + \frac{1}{2}(s - t)\sqrt{-3}, \\ y_3 &= -p - \frac{1}{2}(s + t) - \frac{1}{2}(s - t)\sqrt{-3}, \end{aligned}\right\} \tag{4}$$

and these together with (2) and (3) are the final formulas for the algebraic solution of (1).

When $h^2 - k$ is negative the above formulas also algebraically represent the roots, but the numerical solution is said to fail, or more strictly it should be said that the algebraic formulas fail to give numerical results in as simple forms as desired. In this "irreducible case" the values of s and t are imaginary, but $s+t$ is real and so is $(s-t)\sqrt{-3}$, yet their final numerical values cannot be ascertained by algebraic operations. They can be found graphically by trisecting an angle, or trigonometrically by the use of a table of cosines, but the discussion here is concerned only with algebraic solutions.

II.—The cubic resolvent deduced by Euler, in 1770, for the solution of the quartic equation, is one of the simplest and is hence most frequently quoted. Let the proposed quartic be

$$z^4 + 6Bz^2 + 4Cz + D = 0, \tag{5}$$

and let there be taken the cubic resolvent,

$$y^3 + 3By^2 + \frac{1}{4}(9B^2 - D)y - \frac{1}{4}C^2 = 0, \tag{6}$$

whose roots are y_1, y_2, and y_3. Then the roots of the given quartic are, if C is negative,

$$\left.\begin{aligned} z_1 &= +\sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3}, \\ z_2 &= +\sqrt{y_1} - \sqrt{y_2} - \sqrt{y_3}, \\ z_3 &= -\sqrt{y_1} + \sqrt{y_2} - \sqrt{y_3}, \\ z_4 &= -\sqrt{y_1} - \sqrt{y_2} + \sqrt{y_3}, \end{aligned}\right\} \tag{7}$$

and if C is positive the signs before all the radicals are to be reversed.

The reasoning by which the above is deduced need not be given here; it will be found in many of the mathematical reference books as well as in standard treatises. In applying these formulas, however, to numerical solutions, difficulties are found to arise even when the cubic (6) does not fall under the irreducible case. The numerical example generally given is that stated in the article Algebra in the last edition of the Encyclopædia Britannica, namely $z^4 - 25z^2 + 60z - 36 = 0$, which leads to the cubic $y^3 - \frac{25}{2}y^2 + \frac{769}{16}y - \frac{225}{4} = 0$. The roots of this are found "by the rules for cubics," to be $\frac{9}{4}$, $\frac{16}{4}$ and $\frac{25}{4}$, so that $\sqrt{y_1} = \frac{3}{2}$,

$\sqrt{y_2} = \frac{4}{2}$, $\sqrt{y_3} = \frac{5}{2}$; whence the roots of the quartic are $z_1 = -6$, $z_2 = +3$, $z_3 = +2$, $z_4 = +1$. Now this example has no place in the exemplification of the algebraic solution, for it falls under the irreducible case where the values of the roots of the cubic resolvent cannot be algebraically obtained.

An examination of numerous authorities has been made, but not in a single instance has there been found a numerical example of legitimate algebraic solution by Euler's formulas. The solvable case, numerically, is that where the proposed quartic has two real and two imaginary roots. Such a quartic leads to a cubic resolvent having one real and two imaginary roots, say y_1 real, and y_2 and y_3 imaginary. The square roots of y_2 and y_3 are also imaginary, and hence z_1, z_2, z_3 and z_4 appear in unmanageable imaginary form, although two of these are real quantities. This, apparently, is the reason why legitimate numerical solutions are not found in connection with Euler's resolvent, although this is better adapted than any other resolvent to the exhibition of the roots of the quartic.

III.—In order to discuss the quartic I prefer to use the complete equation

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0. \tag{8}$$

In this, let x be replaced by $z - a$, and it reduces to (5), or,

$$z^4 + 6Bz^2 + 4Cz + D = 0$$

if the coefficients have the values

$$\left.\begin{aligned} B &= -a^2 + b, \\ C &= -2a^3 + 3ab - c, \\ D &= -3a^4 + 6a^2b - 4ac + d. \end{aligned}\right\} \tag{9}$$

Inserting these in Euler's resolvent (6), it becomes (1), namely,

$$y^3 + 3py^2 + 3qy + r = 0,$$

in which p, q and r represent the quantities

$$\left.\begin{aligned} p &= -(a^2 - b), \\ q &= (a^2 - b)^2 + \frac{1}{12}(4ac - 3b^2 - d), \\ r &= -\frac{1}{4}(2a^3 - 3ab + c)^2. \end{aligned}\right| \tag{10}$$

The formulas for the roots of this cubic resolvent are given by (4) and need not be written here again. Then the roots of the quartic are, when $2a^3 - 3ab + c$ is negative,

$$\left.\begin{aligned} x_1 &= -a + \sqrt{y_1} + (\sqrt{y_2} + \sqrt{y_3}), \\ x_2 &= -a + \sqrt{y_1} - (\sqrt{y_2} + \sqrt{y_3}), \\ x_3 &= -a - \sqrt{y_1} + (\sqrt{y_2} - \sqrt{y_3}), \\ x_4 &= -a - \sqrt{y_1} - (\sqrt{y_2} - \sqrt{y_3}), \end{aligned}\right\} \qquad (11)$$

and when $2a^3 - 3ab + c$ is positive the signs before $\sqrt{y_1}$ and the parentheses are to be reversed.

Now to bring these roots into manageable form for numerical solutions let y_1 be regarded as real, and y_2 and y_3 as imaginary, their expressions in terms of s and t being as given in (4). It is clear that both $\sqrt{y_2} + \sqrt{y_3}$ and $\sqrt{y_2} - \sqrt{y_3}$ are real, and in order that their algebraic expressions may be free from imaginaries it will be well to write

$$\left.\begin{aligned} \sqrt{y_2} + \sqrt{y_3} &= \sqrt{y_2 + y_3 + 2\sqrt{y_2 y_3}}, \\ \sqrt{y_2} - \sqrt{y_3} &= \sqrt{y_2 + y_3 - 2\sqrt{y_2 y_3}}. \end{aligned}\right\} \qquad (12)$$

Thus the imaginaries disappear, for as y_2 is of the form $\alpha + \beta i$ and y_3 of the form $\alpha - \beta i$, their sum $y_2 + y_3$ is 2α, and their product $y_1 y_2$ is $\alpha^2 + \beta^2$, both real quantities.

To complete the solution it is now only necessary to obtain the values of y_1, $y_2 + y_3$, $y_2 y_3$, in terms of the coefficients of the given quartic. Let $y_1 = u$, $y_2 + y_3 = v$, and $4y_1 y_2 = w$. Then from (4) are obtained

$$\left.\begin{aligned} y_1 &= -p + s + t = u, \\ y_2 + y_3 &= -2p - s - t = v, \\ 4y_2 y_3 &= (2p + s + t)^2 + 3(s - t)^2 = w, \end{aligned}\right\} \qquad (13)$$

in which $-p$ is $a^2 - b$. The quantities s and t are, by (3),

$$s = (h + \sqrt{h^2 - k})^{\frac{1}{3}}, \qquad t = (h - \sqrt{h^2 - k})^{\frac{1}{3}},$$

and h and k are found by substituting in (2) the values of p, q and r as given by (10), whence

$$\left.\begin{aligned} h &= \frac{1}{8}(a^2 d + b^3 + c^2 - 2abc - bd), \\ k &= \frac{1}{64}(b^2 + \frac{1}{3}d - \frac{4}{3}ac)^3. \end{aligned}\right\} \qquad (14)$$

Thus x_1, x_2, x_3, x_4 are rationally expressed in terms of a, b, c, d by (11), (12), (13), (2) and (14). A further slight simplification results by putting $h = \frac{1}{8}m$ and $k = \frac{1}{64}n$.

IV.—The following, therefore, are final formulas for the algebraic solution of the quartic equation

$$x^4 + 4ax^3 + 6bx^2 + 4cx + d = 0. \tag{15}$$

First, let m and n be determined from

$$\left.\begin{aligned} m &= a^2d + b^3 + c^2 - 2abc - bd, \\ n &= (b^2 + \tfrac{1}{3}d - \tfrac{4}{3}ac)^3. \end{aligned}\right\} \tag{16}$$

Secondly, let s and t be obtained by

$$\left.\begin{aligned} s &= \tfrac{1}{2}(m + \sqrt{m^2 - n})^{\frac{1}{3}}, \\ t &= \tfrac{1}{2}(m - \sqrt{m^2 - n})^{\frac{1}{3}}. \end{aligned}\right\} \tag{17}$$

Thirdly, let u, v and w be found from

$$\left.\begin{aligned} u &= (a^2 - b) + (s + t), \\ v &= 2(a^2 - b) - (s + t), \\ w &= v^2 + 3(s - t)^2. \end{aligned}\right\} \tag{18}$$

Then the four roots of the quartic equation are given by the expressions

$$\left.\begin{aligned} x_1 &= -a + \sqrt{u} + \sqrt{v + \sqrt{w}}, \\ x_2 &= -a + \sqrt{u} - \sqrt{v + \sqrt{w}}, \\ x_3 &= -a - \sqrt{u} + \sqrt{v - \sqrt{w}}, \\ x_4 &= -a - \sqrt{u} - \sqrt{v - \sqrt{w}}, \end{aligned}\right\} \tag{19}$$

in which the signs before the square roots are to be used as written provided $2a^3 - 3ab + c$ is a negative quantity; but if this is positive all radicals except $\sqrt{w}$ are to be reversed in sign.

V.—As a numerical example let the proposed quartic equation be

$$x^4 + 3x^3 + x^2 - 7x - 30 = 0.$$

Here, by comparison with the general form in (15),

$$a = +\frac{3}{4},\quad b = +\frac{1}{6},\quad c = -\frac{7}{4},\quad d = -30.$$

First, by the use (16), are derived the values

$$m = -\frac{226}{27}\quad\text{and}\quad n = -\frac{405224}{729}.$$

Secondly, from (17) are computed,

$$s = +1.2767\quad\text{and}\quad t = -1.6100.$$

Thirdly, by (18) are found

$$u = +0.0625,\quad v = +1.125,\quad w = +26.265.$$

Then, as $2a^3 - 3ab + c$ is a negative number, the formulas (19) furnish

$$x_1 = -0.75 + 0.25 + \sqrt{1.125 + 5.125} = +2,$$
$$x_2 = -0.75 + 0.25 - \sqrt{1.125 + 5.125} = -3,$$
$$x_3 = -0.75 - 0.25 + \sqrt{1.125 - 5.125} = -1 + 2i,$$
$$x_4 = -0.75 - 0.25 + \sqrt{1.125 - 5.125} = -1 - 2i,$$

and each of these exactly satisfies the proposed equation.

As a second example let the given quartic be $x^4 + 7x + 6$. Here $a = 0$, $b = 0$, $c = +\frac{7}{4}$, $d = +6$. First, $m = +3\frac{1}{16}$ and $n = +8$. Secondly, $s = +0.8091$ and $t = +0.6180$. Thirdly, $u = +1.427$, $v = -1.427$, and $w = +2.146$. Then, as c is positive, the formulas (19) furnish the roots

$$x_1 = -1.194 - \sqrt{-1.427 + 1.465} = -1.388,$$
$$x_2 = -1.194 + \sqrt{-1.427 + 1.465} = -1.000,$$
$$x_3 = +1.194 - \sqrt{-1.427 - 1.465} = +1.194 - 1.701i,$$
$$x_4 = +1.194 + \sqrt{-1.427 - 1.465} = +1.194 + 1.701i,$$

and these closely satisfy the proposed quartic.

For the third example let the equation be one with equal roots, $x^4 - 9x^2 + 4x + 12 = 0$. Here $a = 0$, $b = -\frac{3}{2}$, $c = +1$, $d = +12$, whence $m = +\frac{125}{8}$ and $n = \left(\frac{25}{4}\right)^3$, which gives $m^2 - n = 0$, as is always the case for equal roots. Next $s = t = +\frac{5}{4}$, from which $u = +4$, $v = +\frac{1}{2}$, and $w = +\frac{1}{4}$. Then, as c is positive,

$$x_1 = -2 - \sqrt{\frac{1}{2} + \frac{1}{2}} = -3,$$

$$x_2 = -2 + \sqrt{\frac{1}{2} + \frac{1}{2}} = -1,$$

$$x_3 = +2 - \sqrt{\frac{1}{2} - \frac{1}{2}} = +2,$$

$$x_4 = +2 + \sqrt{\frac{1}{2} - \frac{1}{2}} = +2,$$

which are the roots of the given quartic.

Lastly, let the equation be one such that $m = 0$, or $x^4 - 30x^2 - 20x + 20 = 0$. Here, from the given coefficients, m is found to be 0, and $n = \left(\frac{95}{3}\right)^3$. Next, $s + t = 0$ and $3(s - t)^2 = -95$, when $u = 5$, $v = 10$ and $w = 5$. Then, as c is negative, the formulas give the roots

$$x_1 = +\sqrt{5} + \sqrt{10 + \sqrt{5}} = -5.734,$$

$$x_2 = +\sqrt{5} - \sqrt{10 + \sqrt{5}} = +1.262,$$

$$x_3 = -\sqrt{5} + \sqrt{10 - \sqrt{5}} = -0.550,$$

$$x_4 = -\sqrt{5} - \sqrt{10 - \sqrt{5}} = +5.022.$$

VI.—The final formulas (16), (17), (18) and (19) furnish the means for the complete discussion of the circumstances under which the algebraic solution is possible numerically, and also of the conditions necessary for the occurrence of equal roots. The coefficients a, b, c and d are, of course, taken as real quantities in this discussion, although algebraically the formulas are valid for imaginary coefficients also.

First, it is clear that the roots can be obtained numerically, whenever $m^2 - n$ in (17) is a positive quantity, that is, when the coefficients are so related that

$$(a^2d + b^3 + c^2 - 2abc - bd)^2 - (b^2 + \frac{1}{3}d - \frac{4}{3}ac)^3$$

is positive. Then s and t are real, as are also u, v and w, and w is greater than v^2. Hence $v + \sqrt{w}$ will be positive and $v - \sqrt{w}$ will be negative. Accordingly the roots x_1 and x_2 are real and the roots x_3 and x_4 are imaginaries of the form $\alpha \pm \beta i$.

Secondly, the roots of the quartic are easily found whenever $m^2 - n$ is zero, which is well known as the condition for equal roots. For this case s and t are equal, and hence $v = \sqrt{w}$, so that the roots x_3 and x_4 are equal. The formulas (19) then reduce to

$$x_1 = -a + \sqrt{u} + \sqrt{2v},$$
$$x_2 = -a + \sqrt{u} - \sqrt{2v},$$
$$x_3 = x_4 = -a - \sqrt{u},$$

in which $u = a^2 - b + n^{\frac{1}{3}}$ and $v = 2(a^2 - b) - n^{\frac{1}{3}}$, while the signs before the radicals are to be used as before. For two pairs of equal roots the further condition $v = 0$ is necessary, or the coefficients are related so that

$$4(a^2 - b)^3 - (b^3 + \frac{1}{3}d - \frac{4}{3}ac)$$

also vanishes. Then one pair of equal roots is $x_1 = x_2 = -a + \sqrt{3(a^2 - b)}$ and the other is $x_3 = x_4 = -a - \sqrt{3(a^2 - b)}$.

Thirdly, the roots can be computed when $m = 0$, that is when the coefficients are so related that $a^2d + b^3 + c^2 - 2abc - bd = 0$. Then $s + t = 0$, $(s - t)^2 = -n^{\frac{1}{3}}$, and hence $u = a^2 - b$, $v = 2(a^2 - b)$ and $w = v^2 - 3b^2 + d$, whence by (19) the roots are obtained.

The roots of any numerical quartic can be determined when a known relation, other than that expressed by the coefficients, exists between them. For instance, as one of the most important cases, let $x_1x_2 + x_3x_4 = 0$. Then it will be found that the cubic radicals entirely disappear from the general formulas, and that (18) reduce to

$$u = a^2 - \frac{3}{2}b, \quad v = u + a^2, \quad w = 4a(au + c) - d,$$

while the coefficients are connected by the necessary relation $c^2 + ud = 0$. For instance, the quartic $x^4 - 11x^3 + 28x^2 + 36x - 144 = 0$ has its coefficients thus related, and the formulas give the roots $x_1 = +3$, $x_2 = +4$, $x_3 = -6$, and $x_4 = +2$.

When $m^2 - n$ is negative the final formulas lead, in general, to the irreducible case, where the quartic has either four real roots or four imaginary roots. The formulas here correctly represent the roots, but their numerical expressions cannot be algebraically reduced to simple forms.

VII.—In conclusion it may be observed that the algebraic solution of all critical and special cases of the quartic equation is effected by the final formulas (16), (17), (18), and (19), and that simple expressions for the roots may be deduced for some of these cases.

Perhaps the simplest instance is $x^4 - e^4 = 0$, for which, using the formulas in succession, there is found $x_1 = +e$, $x_2 = -e$, $x_3 = +ei$, and $x_4 = -ei$.

The quartic $x^4 + 6bx^2 + d$ is a special case, the solution of which is easily made by quadratics. The general formulas do not, at first sight, appear to reduce to the same expressions, but by observing that $2a^3 - 3ab + c$ vanishes it is clear that $\sqrt{u} = 0$. Hence $s + t = b$, and $v = -3b$. Then, since $st = \frac{1}{4}n^{\frac{1}{3}}$, there is found $3(s-t)^2 = -d$, so that $w = 9b^2 - d$. Therefore (19) reduce to

$$x_1 = x_2 = \pm\sqrt{-3b + \sqrt{9b^2 - d}},$$
$$x_3 = x_4 = \pm\sqrt{-3b - \sqrt{9b^2 - d}},$$

which are the same as given by the quadratic solution.

If $d = 0$ one of the roots of the quartic vanishes and the formulas for the other three will reduce to those for the solution of the cubic. As a special case let $x^4 - e^3x = 0$, where $a = 0$, $b = 0$, $c = -\frac{1}{4}e^3$, $d = 0$. Then (16) give $m = +\frac{1}{16}e^6$, $n = 0$, whence by (17) there is found $s = +\frac{1}{4}e^3$ and $t = 0$. Next (17) give $u = +\frac{1}{4}e^3$, $v = -\frac{1}{2}e^3$ and $w = +\frac{1}{4}e^4$. Finally formulas (18) furnish the roots $x_1 = +e$, $x_2 = 0$, $x_3 = (-\frac{1}{2} + \frac{1}{2}\sqrt{-3})e$; $x_4 = (-\frac{1}{2} - \frac{1}{2}\sqrt{-3})e$, where the coefficients of e are the cube roots of unity.

Thus the final formulas here deduced give the algebraic solution of the quartic equation for all possible instances, and they lead to ready and exact numerical solutions whenever the proposed quartic does not fall under the irreducible case.

LEHIGH UNIVERSITY, *May*, 1892.

A Class of New Theorems on the Number and Arrangement of the Real Branches of Plane Algebraic Curves.

By L. S. Hulburt.

The maximum number of real branches possible to a plane algebraic curve, of order n, and deficiency p, is, as has been shown by Mr. A. Harnack,* $p+1$. He has also proved the existence of such a curve for every deficiency number p. In a recent article on the real branches of algebraic curves,† Mr. David Hilbert of Königsberg has proved, among other important theorems, that the maximum number of *nested branches* possible to a non-singular curve, which has the maximum number of real branches, is $\frac{1}{2}(n-2)$ when n is even, and $\frac{1}{2}(n-3)$ when n is odd. And furthermore he has shown that, for every order number n, there exists a non-singular curve C_n possessing the following remarkable properties:

1). It has the maximum number of real branches, viz. $p+1$ or $\frac{1}{2}(n-1)(n-2)+1$.

2). It has the maximum number of nested branches, $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$, according as n is even or odd.

3). An ellipse can be drawn which shall enclose one or more of the nested branches of C_n and intersect one of the non-nested closed branches in $2n$ real points, whose order of succession shall be the same upon the branch as upon the ellipse.

By *nested branches* is meant a set of closed branches, of which the first is enclosed by each of the others, the second encloses the first and is enclosed by each of the others, and so on.

In proving the existence of such a curve, Hilbert adopts the method employed by Harnack, which consists in deriving from the equation of a known degenerate curve of order n, an equation representing a curve of the same order, which can be shown to possess the required properties.

---

* Mathematische Annalen, Bd. 10. "Ueber die Vieltheiligkeit der ebenen algebraischer Curven."

† Mathematische Annalen, Bd. 38. "Ueber die reellen Züge algebraischer Curven."

It is the object of the present article to extend Hilbert's results to curves having certain singularities, and then to point out the limitations of the method.

I.

Following Hilbert's method, and making use of his results, we shall prove the following

Theorem.—*For every order number n, there exists a curve having* (*a*), *n, or fewer, double points;* (*b*), *the maximum number of real branches, $p+1$; and* (*c*), *the maximum number of nested branches, $\frac{1}{2}(n-2)$ or $\frac{1}{2}(n-3)$ according as n is even or odd.*

We consider first the case where the number of double points is n.

Let $\phi_n = 0$ be the equation of a non-singular curve of order n, possessing the three properties which Hilbert's theorem asserts that such a curve may have. Let $f_2 = 0$ be the equation of the ellipse which meets a non-nested closed branch, b, of the curve $\phi_n = 0$, in $2n$ points, whose order of succession is the same upon b as upon the ellipse. Of these points we select any $n+2$, as $p_1, p_2, \ldots . p_{n+2}$, and connect them by $n+2$ right lines, in such a way that two lines shall pass through each point, and two points shall lie upon each line. Let the product of the equations of these lines be $l_{n+2} = 0$. We form now the equation

$$\psi_{n+2} \equiv \phi_n f_2 \pm \varepsilon l_{n+2} = 0 \qquad \text{(A)}$$

where ε is a variable parameter.

For every sufficiently small value of ε this equation represents a curve of order $n+2$, each point of which lies very near a point of the degenerate curve $\phi_n f_2 = 0$. Now the curves $\phi_n f_2 = 0$ and $l_{n+2} = 0$ have each a double point at each of the points $p_1, p_2, \ldots . p_{n+2}$. Therefore $\psi_{n+2} = 0$ has a double point at each of these points, that is, $n+2$ *double points in all.*

The ellipse $f_2 = 0$ and the closed branch b form, by their intersection, $2n$ regions, each bounded by a single segment of b and a single segment of the ellipse. One of these regions, which we designate by R, contains, in general, some of the nested branches of $\phi_n = 0$. Let s represent that segment of the ellipse which forms a part of the boundary of R. We next choose the sign of ε so that that portion of the curve $\psi_{n+2} = 0$, which lies very near to the segment s, shall lie wholly within R. That this is always possible arises from the fact that the right lines $l_{n+2} = 0$, having already two points each in common with the ellipse, cannot meet s, except at its extremities. Now the form of equation (A) teaches that, if the curve $\psi_{n+2} = 0$ exists within a given region, G, of the plane, then it exists in any other region, H, only when it is possible to pass from G to H along a continuous path, which meets the lines of the figure in an *even* number of points. By virtue of this principle, it follows that, if the curve $\psi_{n+2} = 0$ exists within any one of the $2n$ regions defined above, *and at the same time lies very near that segment of the ellipse which bounds this region*, then it exists within *each* of these regions. But ε has already been so determined that $\psi_{n+2} = 0$ shall exist within R and in the vicinity of s. Therefore it exists within each of the $2n$ regions. Furthermore, if the right lines had no points in common with either the ellipse or b, the curve would have, within each of these regions, a single closed branch, that is, $2n$ such in all. But in the case under consideration, two of these branches are united at each double point, and a little reflection will make clear that every occurrence of a double point reduces the number of these branches by one. Hence *the curve* $\psi_{n+2} = 0$ *has, in the vicinity of the ellipse and* b, $2n - (n + 2)$ *real branches.*

Again, in the immediate neighborhood of each of the remaining $\frac{1}{2}(n-1)(n-2)$ branches of $\phi_n = 0$, there exists a branch of $\psi_{n+2} = 0$. Therefore, *the total number of real branches belonging to* $\psi_{n+2} = 0$ *is* $\frac{1}{2}(n-1)(n-2) + 2n - (n+2) = \frac{1}{2}(\overline{n+2}-1)(\overline{n+2}-2) - (\overline{n+2}) + 1$, *and this is the maximum number.*

The branches of $\psi_{n+2} = 0$ lying in the vicinity of the nested branches of $\phi_n = 0$ are themselves nested branches. That the former curve has yet another nested branch becomes evident on reflecting that the portion of $\psi_{n+2} = 0$, lying very near the segment s of the ellipse, is a portion of a branch, which encloses (or is enclosed by) all the nested branches within the region R, and is, therefore, itself a nested branch. Hence, *the curve* $\psi_{n+2} = 0$ *has* $\frac{1}{2}(n-2) + 1$ *or* $\frac{1}{2}(n-3) + 1$, *that is,* $\frac{1}{2}(\overline{n+2}-2)$ *or* $\frac{1}{2}(\overline{n+2}-3)$ *nested branches, and this is the maximum number.*

Writing n' for $\overline{n+2}$ in the preceding formulas, our argument thus far proves, *for every order number* n', *the existence of a curve* $\psi_{n'} = 0$ *having* n' *double points,* $\frac{1}{2}(n'-1)(n'-2) + 1 = p' + 1$ *real branches, the maximum number, and the maximum number of nested branches,* $\frac{1}{2}(n'-2)$ or $\frac{1}{2}(n'-3)$ *according as* n' *is even, or odd.*

We consider next the case where the number of double points is less than n.

Fixing the attention upon the $2n$ points of intersection of the ellipse with b, we select, not $n+2$ of these points as in the preceding case, but $n+2-r$, and connect them by $n+2-r$ right lines as before. The r remaining right lines are drawn to intersect neither the ellipse nor the branch b. *The curve* $\psi_{n+2} = 0$ *has now only* $n+2-r$ *double points.* It has, then, in the vicinity of the ellipse and b, $2n-(n+2-r)$ real branches. The remaining real branches are, in number and arrangement, the same as before. *The curve has, therefore,* $\frac{1}{2}(n-1)(n-2) + 2n - (n+2-r) = \frac{1}{2}(n'-1)(n'-2) - (n'-r) + 1 = p'+1$ *real branches.* And, finally, it is evident that *it has the maximum number of nested branches.*

In order that our argument may still be valid when $n+2-r < 3$, it is only necessary to draw the two lines, each of which meets the ellipse in but one double point of $\psi_{n+2} = 0$, in such a way that they shall have their second points of intersection with the ellipse upon the *same segment* of the latter.

The preceding results may be summed up as follows:

For every order number n, there exists a curve, $\psi_n = 0$, *having (a)* $n-r$ *double points,* $(r = 0, 1, 2, \ldots n)$, *(b)* $\frac{1}{2}(n-1)(n-2) - (n-r) + 1 = p+1$ *real branches, (c)* $\frac{1}{2}(n-2)$ *or* $\frac{1}{2}(n-3)$ *nested branches, according as n is even or odd.*

II.

Both Harnack and Hilbert, in the articles to which the reference has been made, prove that, for every order number n, there exists a non-singular curve having the maximum number of real branches, $p+1$.

Harnack sets out from a known non-singular curve, $\phi_n = 0$, having $p+1$ real branches, one of which is met by a right line, $f_1 = 0$, in n real points, whose order of succession is the same upon the branch as upon the right line.* He draws, in a suitable manner, $n+1$ other right lines, $l_{n+1} = 0$, and forms the equation

$$\psi_{n+1} \equiv \phi_n f_1 \pm \epsilon l_{n+1} = 0.$$

By suitable choice of the magnitude and sign of ϵ, this new curve will have $n-1$ more real branches than has $\phi_n = 0$, because the latter curve and the right line, $f_1 = 0$, form, by their intersection, n regions, in each of which exists a single closed branch of $\psi_{n+1} = 0$. The latter has, therefore, $\frac{1}{2}(n-1)(n-2) + n = \frac{1}{2}(\overline{n+1}-1)(\overline{n+1}-2) + 1$ real branches, which is the maximum number. The lines $l_{n+1} = 0$ are drawn in such a way that the curve $\psi_{n+1} = 0$ intersects the right line $f_1 = 0$ in $n + 1$ points whose order of succession is the same upon each. Accordingly, the method may be employed to derive, from $\psi_{n+1} = 0$ and $f_1 = 0$, a new curve, $\psi_{n+2} = 0$, possessing the same properties. In this manner Harnack proceeds, step by step, from a known curve with $p + 1$ real branches to one of any order having the maximum number of real branches.

---

* A non-singular cubic possesses all these properties.

As we have already seen, Hilbert's method differs from this merely in the use of a conic, $f_2 = 0$, as the auxiliary curve, in place of the right line $f_1 = 0$. The derived curve is of order $n + 2$, has $2n - 1$ more real branches than had the original curve $\phi_n = 0$, and, hence, has the maximum number of real branches.

The availability of these two methods of proof is due to the fact that the number of new branches formed is, in either case, exactly sufficient to maintain at the maximum the number of real branches of the derived curves.

Now it is worthy of note, and this is the point to which particular attention is called, that, if any curve other than the right line or the conic, be chosen as the auxiliary curve, the derived curve will have less than the maximum number of real branches.

For, let $f_k = 0$ represent a curve of order k intersecting a single branch of the non-singular curve $\phi_n = 0$ in kn points, whose order of succession is the same upon the one curve as upon the other. Then the equation

$$\psi_{n+k} \equiv \phi_n f_k \pm \varepsilon l_{n+k} = 0$$

represents a curve of order $n + k$, and under proper choice of ε, and suitable convention concerning the position of the right lines $l_{n+k} = 0$, this derived curve will have a single closed branch in each of the kn regions formed by the intersection of $\phi_n = 0$ with $f_k = 0$. If now we assume that $\phi_n = 0$ has $p + 1$ real branches, the total number of real branches belonging to $\psi_{n+k} = 0$ is $\frac{1}{2}(n - 1)(n - 2) + kn$. Let us now assume that this is the maximum number of real branches possible to $\psi_{n+k} = 0$. This assumption gives, for the determination of k, the equation

$$\tfrac{1}{2}(n + k - 1)(n + k - 2) + 1 = \tfrac{1}{2}(n - 1)(n - 2) + kn.$$

This is a quadratic in k, whose roots are 1 and 2.

Hence *the auxiliary curve, $f_k = 0$ must be either a right line or a conic*, and this was to be proved.

CLARK UNIVERSITY, WORCESTER, MASS., *April* 28, 1892.

The Symbolic Notation of Aronhold and Clebsch.

BY WILLIAM F. OSGOOD, *Cambridge, Mass.*

In the 55th volume of Crelle, Aronhold communicated a symbolic notation for functional invariants of a general linear transformation, and applied this notation with marked success to the investigation of the ternary cubic. Clebsch then showed, in Crelle 58, p. 117 and ib. 59, and later in the Abh. d. k. Ges. d. Wiss. zu Göttingen, Bd. 17, that this notation suffices for the symbolic representation, in perspicuous form, of any functional invariant of simultaneous forms. The right to define functional invariants by means of certain symbolic expressions, well adapted to display their essential characteristics, was thus established.

This notation has been adopted generally among German mathematicians. As yet it has not, however, met with the acceptance in English literature to which its merits entitle it. The writer believes that this is due, in part, to the lack of an easily accessible and systematic exposition of the notation and of those fundamental properties of the same that render it so well adapted to the expression of functional invariants. The object of the present paper is to supply such an exposition.

§1.—*The Symbolic Notation for Binary Forms.*

Let the binary form of degree n be written

$$f_n(x) = \bar{a}_0 x_1^n + n_1 \bar{a}_1 x_1^{n-1} x_2 + n_2 \bar{a}_2 x_1^{n-2} x_2^2 + \dots .$$

where

$$n_k = \frac{n!}{(n-k)!\, k!}$$

This expression resembles the expansion of the binomial

$$(a_1 x_1 + a_2 x_2)^n = a_1^n x_1^n + n_1 a_1^{n-1} a_2 x_1^{n-1} x_2 + n_2 a_1^{n-2} a_2^2 x_1^{n-2} x_2^2 + \dots$$

in its homogeneity in x, in the number of its terms and the degree of the latter in x_1 and x_2, and in its numerical coefficients. The $n+1$ coefficients $\bar{a}$ of $f_n(x)$

are, therefore, completely characterized by the $n+1$ products of powers of a_1 and a_2, and we may write *symbolically*

$$\bar{a}_0 = a_1^n, \quad \bar{a}_1 = a_1^{n-1}a_2, \quad \bar{a}_2 = a_1^{n-2}a_2^2, \ \ldots. \quad \bar{a}_k = a_1^{n-k}a_2^k, \ \ldots. \ ;$$
$$f_n(x) = (a_1x_1 + a_2x_2)^n = a_x^n,$$

where a_x is an abbreviation for $a_1x_1 + a_2x_2$. Thus in any expression in which the $\bar{a}$'s enter linearly, they may be replaced by their symbolic representatives in the a's, for these latter symbolic products are linearly independent of each other. The symbols a_1, a_2, taken by themselves, have no meaning in terms of the coefficients $\bar{a}$; only when combined in expressions of degree n in a_1, a_2 are they capable of interpretation in terms of the $\bar{a}$'s.

A simple example of the application of this notation is the following. Let x be replaced by x' by means of the linear transformation

$$\left.\begin{array}{l} x_1 = \xi_1x_1' + \eta_1x_2' \\ x_2 = \xi_2x_1' + \eta_2x_2' \end{array}\right\} \quad r = \begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix} \qquad (1)$$

and let the form into which $f_n(x)$ is transformed be written

$$f_n'(x') = \bar{a}_0'x_1'^n + n_1\bar{a}_1'x_1'^{n-1}x_2' + n_2\bar{a}_2'x_1'^{n-2}x_2'^2 + \ldots.$$

Then the coefficients $\bar{a}'$ are linear functions of the coefficients $\bar{a}$ and are homogeneous functions of degree n in the coefficients of the transformation ξ, η, for $f_n'(x') \equiv f_n(x)$. The relation between $\bar{a}'$ and $\bar{a}$ can be expressed symbolically in simple form. Let

$$f_n'(x') = (a_1'x_1' + a_2'x_2')^n = a_{x'}'^n,$$

so that $\quad \bar{a}_0' = a_1'^n, \quad \bar{a}_1' = a_1'^{n-1}a_2', \ \ldots. \quad \bar{a}_k' = a_1'^{n-k}a_2'^k.$

On the other hand,

$$a_1x_1 + a_2x_2 = a_1(\xi_1x_1' + \eta_1x_2') + a_2(\xi_2x_1' + \eta_2x_2') = a_\xi x_1' + a_\eta x_2',$$

and thus we have two symbolic expressions for $f_n'(x')$, namely $(a_1'x_1' + a_2'x_2')^n$ and $(a_\xi x_1' + a_\eta x_2')^n$. Hence we may put

$$a_1' = a_\xi, \ a_2' = a_\eta, \qquad (2)$$

and thus the coefficients $\bar{a}'$ can be represented symbolically in the simple form:

$$\bar{a}_0' = a_\xi^n, \quad \bar{a}_1' = a_\xi^{n-1}a_\eta, \quad \bar{a}_2' = a_\xi^{n-2}a_\eta, \ \ldots. \quad \bar{a}_k' = a_\xi^{n-k}a_\eta^k \ \ldots.$$

The conciseness and clearness of expression that this example so well illustrates are characteristic features of this symbolic notation.

When it is desired to represent symbolically a function of the $\bar{a}$'s of higher degree than the first, confusion would arise from the direct introduction of the a's, since, for example, $a_1^n a_2^n = a_1^{n-i} a_2^i \, . \, a_1^i a_2^{n-i}$ would represent equally well any one of the products $\bar{a}_i \bar{a}_{n-i}$ $(i = 0, 1, 2, \dots .)$. In order to avoid this ambiguity, further symbols b, c, ... are introduced, subject to the same conditions as the a's:

$$\bar{a}_k = a_1^{n-k} a_2^k = b_1^{n-k} b_2^k = c_1^{n-k} c_2^k = \dots ,$$

$$f_n(x) = a_x^n = b_x^n = c_x^n = \dots .$$

and, in a product of $\bar{a}$'s, each $\bar{a}$ is replaced by a different symbol. Thus $\bar{a}_1^2 \bar{a}_4 = a_1^{n-1} a_2 \, . \, b_1^{n-1} b_2 \, . \, c_1^{n-4} c_2^4 = a_1^{n-1} a_2 \, . \, c_1^{n-1} c_2 \, . \, b_1^{n-4} b_2^4 = \dots .$ If we agree, therefore, to introduce symbols equal in number to the degree in the $\bar{a}$'s of the product we wish to represent, an ambiguity is no longer possible.

Suppose, for example, it be required to represent symbolically the discriminant Δ of the binary quadratic form

$$f_2(x) = \bar{a}_0 x_1^2 + 2\bar{a}_1 x_1 x_2 + \bar{a}_2 x_2^2 = a_x^2 = b_x^2 = c_x^2 = \dots .$$

We have
$$\Delta = \begin{vmatrix} \bar{a}_0 & \bar{a}_1 \\ \bar{a}_1 & a_2 \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1 a_2 \\ b_1 b_2 & b_2^2 \end{vmatrix} = a_1 b_2 (ab),$$

where $(ab) = | a_1 b_2 |$. But Δ might equally well have been expressed symbolically in the form:

$$\Delta = \begin{vmatrix} b_1^2 & b_1 b_2 \\ a_1 a_2 & a_2^2 \end{vmatrix} = b_1 a_2 (ba) = - b_1 a_2 (ab).$$

Adding these two expressions for Δ together, we have

$$2\Delta = (a_1 b_2 - a_2 b_1)(ab) = (ab)^2, \qquad \Delta = \tfrac{1}{2}(ab)^2.$$

From the symbolic expression thus obtained the invariant character of Δ with regard to linear transformation is at once evident. For, (1) and (2):

$$(a'b') = \begin{vmatrix} a_1\xi_1 + a_2\xi_2 & a_1\eta_1 + a_2\eta_2 \\ b_1\xi_1 + b_2\xi_2 & b_1\eta_1 + b_2\eta_2 \end{vmatrix} = r(ab),$$

and hence
$$(a'b')^2 = r^2 (ab)^2$$
$$\Delta' = r^2 \Delta .$$

Here, again, the simplicity and perspicuity of this notation are striking features. One sees at once, when Δ is written in the symbolic form, 1) that

Δ is an invariant, 2) that the power of r that enters is the second, 3) that the degree of Δ in the coefficients of f is the second (since Δ contains two symbols).*

§2.—*The Symbolic Expression of Invariants of Binary Forms.*

From the purely formal relations between the symbols

$$\left.\begin{aligned} a'_{x'} &= a_x \\ (a'b') &= r(ab) \end{aligned}\right\} \tag{3}$$

it is at once evident that invariants can be written down at pleasure. Thus

$$(ab)^{2i}a_x^{n-2i}b_x^{n-2i}, \qquad (ab)^2(bc)^2(ac)^2a_x^{n-4}b_x^{n-4}c_x^{n-4}, \qquad (ab)^n$$

are certainly invariants (I include here under this expression covariants as well). The last one vanishes identically when n is odd, for then, by an interchange of the equivalent symbols a and b,

$$(ab)^n = (ba)^n = -(ab)^n, \quad 2(ab)^n = 0,$$

and it is shown in like manner that any invariant that changes sign when equivalent symbols are interchanged vanishes identically, for example

$$(ab)^{2i+1}a_x^{n-2i-1}b_x^{n-2i-1} \equiv 0.$$

To obtain an invariant, therefore, it is only necessary to write down a product of factors of the above types (3)

$$\Pi\{(ab),\ (bc),\ \ldots.\ a_x,\ b_x,\ \ldots.\}$$

subject to the condition that each symbol a, b, ... shall appear just n times. The number, λ, of the determinant factors in Π will be the exponent of r in the invariant equation

$$\Pi' = \Pi\{(a'b'),\ (b'c'),\ \ldots.\ a'_{x'},\ b'_{x'},\ \ldots.\} = r^\lambda\Pi\{(ab),\ (bc)\ \ldots.a_x,\ b_x\ \ldots.\}$$

Furthermore, if Π_1, Π_2, Π_3, ... are such products, all having the same number of symbols of f, the same degree in x, and the same λ, then

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \ldots.$$

where $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, ... are arbitrary numerical constants, will also be an invariant, $P' = r^\lambda P$.

---

* For a further treatment of the subject of this paragraph see Gordan's *Vorlesungen über Invariantentheorie*, edited by Kerschensteiner, Leipzig, 1885, Vol. II, where this symbolic notation is used throughout; Clebsch's *Theorie der algebraischen binären Formen*, Leipzig, 1872; Clebsch's *Vorlesungen über Geometrie*, edited by Lindemann, Leipzig, 1876, Vol. I, p. 167.

Thus far we have considered only a single form $f_n(x)$. The extension to the case of simultaneous forms $f_n(x)$, $\phi_m(x)$, ... presents no difficulty. Let $\phi_m(x) = \alpha_x^m = \beta_x^m = \ldots.$ Then in the symbolic product

$$\Pi\{(ab)\ldots., (a\alpha)\ldots., (\alpha\beta)\ldots., a_x\ldots., \alpha_x\ldots.\}$$

the symbols $a, b, \ldots.$ must each appear n times, the symbols $\alpha, \beta, \ldots.$ m times, etc. Π will then be a simultaneous invariant of $f, \phi, \ldots.$, and

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \ldots.,$$

formed subject to conditions similar to the previous, will also be a simultaneous invariant of $f, \phi, \ldots.$

We have thus seen how to form at pleasure, with the aid of this symbolic notation, invariants of a form $f_n(x)$ or of simultaneous forms $f_n(x)$, $\phi_m(x)$, ... , and the form of the expressions thus obtained was such that the essential properties of the functions of the coefficients of $f, \phi, \ldots.$ that they represented were evident at the first glance. Now the fundamental property of this notation,—the property that renders it an adequate and well adapted means of investigation for the theory of invariants,—is the converse of the above, namely, that every invariant can be expressed in the form

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \ldots.$$

Stated as a theorem*: *Every rational integral invariant* (including covariants) *of the binary forms* $f_n(x) = a_x^n = b_x^n = \ldots.$, $\phi_m(x) = \alpha_x^m = \beta_x^m = \ldots.$, *etc., can be expressed as a rational integral function of factors* $(ab)\ldots.$, $(a\alpha)\ldots.$, $(\alpha\beta)\ldots.$, $a_x\ldots.$, $\alpha_x\ldots.$

§3.—*The Symbolic Notation for Ternary Forms.*

The extension of the foregoing symbolic notation to ternary forms is at once evident. Let

$$f_n(x) = \bar{a}_{n,0,0}x_1^n + n\bar{a}_{n-1,1,0}x_1^{n-1}x_2 + n(n-1)\bar{a}_{n-2,1,1}x_1^{n-2}x_2x_3 + \ldots. \\ + \frac{n!}{i!\,j!\,k!}\bar{a}_{ijk}x_1^ix_2^jx_3^k\ldots.,$$

where $i + j + k = n$, be a ternary form of the n^{th} degree. Comparing it with the expansion of the trinomial

---

* An elementary proof of this theorem is to be found in Clebsch-Lindemann's *Geometrie*, p. 184. Other proofs are given in Clebsch's and Gordan's books previously referred to.

$$a_x^n = (a_1x_1 + a_2x_2 + a_3x_3)^n = a_1^n x_1^n + n a_1^{n-1} a_2 x_1^{n-1} x_2 + n(n-1) a_1^{n-2} a_2 a_3 x_1^{n-2} x_1 x_2 + \ldots .$$
$$+ \frac{n!}{i!\ j!\ k!} a_1^i a_2^j a_3^k x_1^i x_2^j x_3^k + \ldots .,$$

where $i + j + k = n$ and a_x is written as an abbreviation for $a_1x_1 + a_2x_2 + a_3x_3$, we see that the coefficients $\bar{a}$ are completely characterized by the products of the a's, and we may, therefore, write symbolically

$$\bar{a}_{n,0,0} = a_1^n, \qquad \bar{a}_{n-1,1,0} = a_1^{n-1} a_2, \ldots . \qquad \bar{a}_{ijk} = a_1^i a_2^j a_3^k \ldots .$$

It is shown, similarly as before, that if, when a linear transformation of the x's is made:

$$x_i = \xi_i x_1' + \eta_i x_2' + \zeta_i x_3', \quad (i = 1, 2, 3), \quad r = |\xi_1 \eta_2 \zeta_3|,$$

the form, into which $f_n(x)$ goes, is written

$$f_n'(x') = \sum \frac{n!}{i!\ j!\ k!} a_{ijk}' x_1'^i x_2'^j x_3'^k = (a_1'x_1' + a_2'x_2' + a_3'x_3')^n = a_{x'}'^n,$$

then $$a_1' = a_\xi,\ a_2' = a_\eta,\ a_3' = a_\zeta$$

and $$\bar{a}_{ijk}' = a_\xi^i a_\eta^j a_\zeta^k.$$

When the function of the $\bar{a}$'s that is to be expressed symbolically is of higher degree than the first, additional symbols must be introduced. Thus if Δ is the discriminant of the quadratic

$$f_2(x) = \bar{a}_{200}x_1^2 + \bar{a}_{020}x_2^2 + \bar{a}_{002}x_3^2 + 2\bar{a}_{011}x_2x_3 + 2\bar{a}_{101}x_3x_1 + 2\bar{a}_{110}x_1x_2 = a_x = b_x = c_x \ldots .$$

we have $$\Delta = \begin{vmatrix} \bar{a}_{200} & \bar{a}_{110} & \bar{a}_{101} \\ \bar{a}_{110} & \bar{a}_{020} & \bar{a}_{011} \\ \bar{a}_{101} & \bar{a}_{011} & \bar{a}_{002} \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ c_1c_3 & c_2c_3 & c_3^2 \end{vmatrix} = a_1b_2c_3(abc),$$

where (abc) denotes the determinant $|a_1b_2c_3|$. By interchanging the symbols a, b, c, and adding together the 6 expressions for Δ thus arising, we have

$$6\Delta = (abc)[a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3] = (abc)^2,$$
$$\Delta = \tfrac{1}{6}(abc)^2.$$

Let ρ, σ, τ be symbols of any form or forms,

$$f = a_x^n = b_x^n \ldots ., \quad \phi = \alpha_x^m = \beta_x^m = \ldots ., \text{ etc.},$$

ρ', σ', τ' symbols of the corresponding transformed forms f', ϕ', etc., so that $\rho_1' = \rho_\xi$, $\rho_2' = \rho_\eta$, $\rho_3' = \rho_\zeta$, etc. Then the law for the multiplication of determinants gives the relation

$$(\rho'\sigma'\tau') = r\,(\rho\sigma\tau).$$

From this it follows that

$$(a'b'c')^2 = r^2(abc)^2, \qquad \Delta' = r^2\Delta.$$

Thus far the analogy between the binary and the ternary forms has been a perfect one. But whereas, in the case of the former, it was necessary to consider only functions of a single variable, x, it is now necessary to consider, in addition, functions of u, where the u's may be interpreted geometrically as the coordinates of a line, the x's being taken as the coordinates of a point. The corresponding transformations of x and u are, since $u'_{x'} = u_x$,

$$\left.\begin{array}{l} x_1 = \xi_1 x'_1 + \eta_1 x'_2 + \zeta_1 x'_3 \\ x_2 = \xi_2 x'_1 + \eta_2 x'_2 + \zeta_2 x'_3 \\ x_3 = \xi_3 x'_1 + \eta_3 x'_2 + \zeta_3 x'_3 \end{array}\right\}, \quad \left.\begin{array}{l} u'_1 = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 = u_\xi \\ u'_2 = \eta_1 u_1 + \eta_2 u_2 + \eta_3 u_3 = u_\eta \\ u'_3 = \zeta_1 u_1 + \zeta_2 u_2 + \zeta_3 u_3 = u_\zeta \end{array}\right\}, \quad r = |\xi_1\eta_2\zeta_3|$$

Hence it appears that, when x is linearly transformed, the line coordinates u and the symbols a, b, c are subjected to the same transformation. From this it follows that, whatever symbols ρ, σ may be,

$$(\rho'\sigma'u') = r(\rho\sigma u).$$

§4.—*The Symbolic Expression of Invariants of Ternary Forms.*

From the purely formal relations that have been obtained, namely,

$$\left.\begin{array}{r} \rho'_{x'} = \rho_x \\ (\rho'\sigma'\tau') = r(\rho\sigma\tau) \\ (\rho'\sigma'u') = r(\rho\sigma u) \end{array}\right\} \tag{4}$$

and with the help of the identical invariant u_x, invariants of the form f can be formed at pleasure. Thus for the cubic (cf. Clebsch-Lindemann, *Geometrie*, pp. 543 and 553),

$$f_3(x) = a_x^3 = b_x^3 = c_x^3 = \ldots\ldots,$$
$$S = (abc)(abd)(acd)(bcd),$$

is the invariant, whose vanishing is the condition that the Hessian of f,

$$\Delta = (abc)^2 a_x b_x c_x = 0,$$

break up into three straight lines, while

$$\Theta = (abu)^2 a_x b_x = 0$$

is the equation in line coordinates of the first polar of x, when x is taken as constant, or the locus of the points x, whose first polars touch a given line u, when u is taken as constant. The equation of f itself in line coordinates is

$$(abu)^2(cdu)^2(acu)(bdu) = 0 .$$

In general, if

$$f = a_x^n = b_x^n = \ldots\ldots, \qquad \phi = \alpha_x^m = \beta_x^m = \ldots\ldots, \qquad \psi = \text{etc.},$$

be any simultaneous forms, an invariant of these forms can be obtained by multiplying together factors of the type (4) and the identical invariant u_x:

$$\Pi\{(abc)\ldots\ldots(a\alpha\beta)\ldots\ldots(\alpha\beta\gamma)\ldots\ldots(abu)\ldots\ldots(a\alpha u)\ldots\ldots(\alpha\beta u)\ldots\ldots a_x\ldots\ldots\alpha_x\ldots\ldots u_x\},$$

care being taken that the symbols $a, b, \ldots\ldots$ appear just n times, the symbols $\alpha, \beta, \ldots\ldots$ just m times, etc. The number, λ, of determinant factors will be the exponent of r in the invariant equation

$$\Pi' = r^\lambda \Pi .$$

When there is only one form f, Π reduces to

$$\Pi\{(abc)\ldots\ldots(abu)\ldots\ldots a_x\ldots\ldots u_x\}.$$

If $\Pi_1, \Pi_2, \Pi_3 \ldots\ldots$ are such products, all having the same number of symbols of f, of ϕ, etc., the same degree in u and in x, and the same λ, then

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 \ldots\ldots,$$

where the $\varkappa$'s are arbitrary numerical constants, is also a simultaneous invariant of the forms f, $\phi \ldots\ldots$

Thus for the ternary as well as for the binary forms we have a symbolic notation, with the aid of which we can obtain invariants at pleasure, and the expression of the same is one that renders their chief characteristics, namely, their degree in the coefficients of each of the ground forms (shown by the number of symbols of each form that appear), in the point coordinates x, in the line coordinates u, and the value of λ (shown by the number of determinant factors), evident at the first glance. And just as in the case of the binary forms, so here, the converse is also true, namely, that every invariant of f can be expressed in the form*

$$P = \varkappa_1\Pi_1 + \varkappa_2\Pi_2 + \varkappa_3\Pi_3 + \ldots\ldots$$

---

* See the memoirs by Clebsch cited at the beginning of this paper; also Cl.-Lind., *Geom.*, p. 270. An elegant proof of this theorem is given in Gordan-Kerschensteiner, *Invariantentheorie*, II, §9, p. 110. The corresponding theorem for binary forms is first proved, and this proof is then extended to the case of n-ary forms.

Every rational integral invariant (including covariants, etc.) *of the ternary forms* $f = a_x^n = b_x^n = \ldots\ldots, \phi = \alpha_x^m = \beta_x^m \ldots\ldots, \psi =$ *etc., can be expressed as a rational integral function of the identical invariant,* u_x, *and of factors of the type* $(\rho\sigma\tau)$, $(\rho\sigma u)$, ρ_x, *where* ρ, σ, τ *are symbols of* f, ϕ, ...

§5.—*The Symbolic Representation of Higher Forms.*

The application of this symbolic notation to the representation of higher forms presents no difficulty. Thus it is evident that the biternary form can be written

$$(a_1x_1 + a_2x_2 + a_3x_3)^n(u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3)^m = a_x^n u_\alpha^m,$$

and the invariants of this form will be rational integral functions of factors of the type*

$$a_x; \quad (abu), \quad (abc), \quad a_\alpha, \quad u_\alpha, \quad (\alpha\beta x), \quad (\alpha\beta\gamma),$$

and of the identical invariant u_x. In the memoir in the Göttingen *Abhandlungen* above cited,† Clebsch proves that a ground form with several cogredient variables of each class (the number of the homogeneous point coordinates being n) can always be replaced by a system of ground forms (das reducirte äquivalente System), each containing at most but one variable from each class and such that the totality of the invariants of this system coincides with the totality of the invariants of the given ground form. Any simultaneous rational integral invariant of the forms of this system, and hence of the ground form itself, is expressible symbolically, rationally and integrally in terms of determinant factors of a few fundamental types, and thus Aronhold's symbolic notation is shown to have the same fundamental properties in the case of the higher forms as for the binary and ternary forms.‡

§6.—*On Certain Identical Relations.*

If, in the identically vanishing determinant,

$$\begin{vmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} \equiv 0,$$

---

* Cf. Cl.-Lind., *Géom.*, p. 942.

† Cf. also Cl.-Lind., *Geom.*, pp. 269 and 924.

‡ See also the reference to Gordan-Kerschensteiner given at the end of §4.

the first column, multiplied by x_1, and the second column, multiplied by x_2, be added to the third, the determinant thus arising can be expanded in the following form:

$$\begin{vmatrix} a_1 & a_2 & a_x \\ b_1 & b_2 & b_x \\ c_1 & c_2 & c_x \end{vmatrix} \equiv (ab)c_x + (bc)a_x + (ca)b_x \equiv 0 . \qquad \text{I}$$

If, in this identity, we put $x_1 = d_2$, $x_2 = -d_1$, we have

$$(ab)(cd) + (bc)(ad) + (ca)(bd) \equiv 0 . \qquad \text{II}$$

Again, if $c_1 = y_2$, $c_2 = -y_1$, I assumes the form

$$a_x b_y - a_y b_x = (ab)(xy) . \qquad \text{III}$$

Similar identities exist for the ternary forms. The development of

$$0 \equiv \begin{vmatrix} a_1 & a_2 & a_3 & 0 \\ b_1 & b_2 & b_3 & 0 \\ c_1 & c_2 & c_3 & 0 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_x \\ b_1 & b_2 & b_3 & b_x \\ c_1 & c_2 & c_3 & c_x \\ d_1 & d_2 & d_3 & d_x \end{vmatrix}$$

gives

$$(abc)d_x - (bcd)a_x + (cda)b_x - (dab)c_x \equiv 0 , \qquad \text{IV}$$

and from this follows, by putting

$$x_1 = (e_2 f_3 - e_3 f_2) = (ef)_1, \quad x_2 = (e_3 f_1 - e_1 f_3) = (ef)_2, \quad x_3 = (e_1 f_2 - e_2 f_1) = (ef)_3,$$

$$(abc)(def) - (bcd)(aef) + (cda)(bef) - (dab)(cef) \equiv 0 . \qquad \text{V}$$

The determinant

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

can be developed into two expressions, which, put equal to each other, give the identity

$$a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_x b_z c_y - a_z b_y c_x - a_y b_x c_z = (abc)(xyz) . \qquad \text{VI}$$

Finally, the identities

$$a_x b_y - a_y b_x = (abu), \qquad \text{VII}$$

where

$$u_1 = (x_2 y_3 - x_3 y_2) = (xy)_1, \quad u_2 = (x_3 y_1 - x_1 y_3) = (xy)_2,$$

$$u_3 = (x_1 y_2 - x_2 y_1) = (xy)_3,$$

and $$a_x b_y - a_y b_x = (xyz)\,, \qquad \text{VIII}$$

where $$z_1 = (ab)_1\,, \quad z_2 = (ab)_2\,, \quad z_3 = (ab)_3\,,$$

can be verified by actually multiplying out.

These identities are of prime importance in transforming symbolic expressions.*

CAMBRIDGE, MASS., *March*, 1892.

---

* The significance of these identities for binary forms is brought out in Gordan-Kerschensteiner, *Invariantentheorie*, II, Nos. 12 and 117, where it is shown that every relation between symbolic products (and hence, for example, between the invariants of a form or forms) can be referred directly to these identities. And the significance of these identities for ternary forms is no less fundamental.

The System of Two Simultaneous Ternary Quadratic Forms.

By William F. Osgood, *Cambridge, Mass.*

Prof. Gordan has shown that all invariants and covariants of a single binary form or of simultaneous binary forms can be expressed as rational, integral functions of a certain set of invariants and covariants, finite in number, of the ground form or forms.[*] This set he calls the "*system*" of the ground forms. An essential advantage of the method that he employs is that it also furnishes, simultaneously with the proof of the theorem, the system itself; for he actually establishes a set of invariants (I include under "invariants" in this article covariants, contravariants, etc.), finite in number and having the property that every invariant can be expressed rationally and integrally in terms of these invariants. This set may contain superfluous members, namely, members that are themselves expressible rationally and integrally in terms of other members of the set. Such members are to be discarded, and the remaining members then constitute the system of the ground forms.

The method employed for the binary forms Gordan then extended to the ternary forms.[†] He discovered the characteristic marks of a set of invariants that form the system and showed how actually to find such a set. It may be remarked that there is a certain degree of choice as to the particular invariants that shall be taken to form the system. That this system would always be finite he did not prove generally, but he examined a number of special cases,[‡] in all of which he found that the system is finite. Thus he established the system of the ternary cubic and biquadratic, and showed further, both for ternary and for quaternary forms, that if any number of ground forms each have finite systems, the system of these forms, considered as simultaneous, is also finite.

---

[*] *Ueber das Formensystem binärer Formen*, Leipzig, 1875, in which the earlier demonstrations given in Crelle, 69, and Math. Ann., 8, are revised and simplified. See also Gordan-Kerschensteiner, *Invariantentheorie*, II, Leipzig, 1887, and Clebsch, *Binäre Formen*, Leipzig, 1872.

[†] *Ueber ternäre Formen dritten Grades*, Math. Ann., Bd. 1.

[‡] *Ueber das Formensystem*, etc., p. 50. See also Math. Ann., Bd. 17, p. 217.

Recently, Dr. Hilbert* has proved the general theorem that the system of an n-ary form or of such simultaneous forms is finite. How to find this system is, however, a question that at present has not been answered.

The object of the present paper is to show, in an elementary manner, how Gordan established the system of two simultaneous ternary quadratic forms. An account of this method is given in Clebsch-Lindemann, *Geometrie*, I, Leipzig, 1876, p. 288, but for the reader that is unacquainted with Gordan's methods this account is too brief to be easily intelligible. The importance of these investigations, which at once form a natural introduction to the theory of the ternary forms and help to systematize the geometric theory of a system of two conics, renders it desirable that they should be easily accessible. The notation employed is the symbolic notation of Aronhold and Clebsch, the essential properties of which I have explained in the present number of this Journal (*The Symbolic Notation of Aronhold and Clebsch*).

§1.—*The Symbolic Representation of Invariants of f and f'.*

Let the two simultaneous ternary quadratic forms be

$$\left.\begin{aligned} & f = a_x^2 = b_x^2 = c_x^2 = \dots\dots \\ \text{and} \qquad & f' = a_x'^2 = b_x'^2 = c_x'^2 = \dots\dots \end{aligned}\right\} \qquad (1)$$

Then surely the discriminants†

$$A_{111} = (abc)^2, \quad A_{222} = (a'b'c')^2$$

of f and f', as also the forms in u which, put equal to 0, are the equations in line coordinates of the conics $f = 0$, $f' = 0$:

$$F_{11} = (abu)^2, \quad F_{22} = (a'b'u)^2$$

must belong to the system.‡

---

* *Ueber die Theorie der algebraischen Formen*, Math. Ann., Bd. 36.

† Cf. *Symb. Not.* §3.

‡ See Clebsch-Lindemann, *Geometrie*, I, p. 287, where it is shown that the system of a single ternary quadratic form $f = a_x^2 = b_x^2 = \dots\dots$ consists of the form itself, the discriminant $(abc)^2$ and the form $(abu)^2$, the vanishing of which gives the equation of the conic $f = 0$ in line coordinates. The identical invariant u_x belongs to every system and, therefore, is not further explicitly mentioned.

Whenever two symbols of f, as a and b, occur in the combination

$$(a_2b_3 - a_3b_2) = (ab)_1, \quad (a_3b_1 - a_1b_3) = (ab)_2, \quad (a_1b_2 - a_2b_1) = (ab)_3,$$

it is of practical advantage to replace them by a single symbol α:

$$\alpha_1 = (ab)_1, \quad \alpha_2 = (ab)_2, \quad \alpha_3 = (ab)_3,$$

or, abbreviated, $\alpha = (ab)$. Thus

$$(abu) = u_1\alpha_1 + u_2\alpha_2 + u_3\alpha_3 = u_\alpha.$$

The symbols a, b are cogredient with the line coordinates u when the linear transformation

$$x_i = \xi_i x_1' + \eta_i x_2' + \zeta_i x_3', \quad i = 1,\ 2,\ 3$$

is performed,* but the combinations (ab) of a and b, *i. e.* the symbols α, are cogredient with the point coordinates x, and hence the notation u_α, which suggests an analogy between α and x, is justified. We now have

$$A_{111} = a_\alpha^2, \quad A_{222} = a'^2_{\alpha'}, \quad F_{11} = u_\alpha^2, \quad F_{22} = u_{\alpha'}^2, \qquad (2)$$

where, in the first expression, $\alpha = (bc)$.

The simplification effected by the introduction of the symbols α, β, ... appears from the following theorem: *if the symbols* a, b *appear once in the symbolic product* Π *in the combination* (ab):

$$\Pi = (abv)M = v_\alpha M,$$

where v *is any element cogredient with* u, *then their further appearance will also be in the combination* (ab), *so that*

$$\Pi = v_\alpha w_\alpha M',$$

where M' *no longer contains* a, b, *and hence they can be completely replaced by the single symbol* α. For either Π is already of the form demanded by the theorem, or else:

$$\Pi = (abv)a_s b_t M_1,$$

where a, b now enter only explicitly. (s, t are, for example, (cd), $(c'd)$, $(c'd')$, (cu), $(c'u)$, etc., or x; in general, they are any combinations of symbols and coordinates that are cogredient with x. In the same way, v, w are symbols c, d, a', b', or u, or any combinations of symbols and coordinates that are cogredient with u.) When a, b are interchanged, the value of Π is unaltered, the form of Π becomes:

$$\Pi = (bav)b_s a_t M_1 = -(abv)a_t b_s M_1.$$

---

* Cf. *Symb. Not.* §8.

Adding these two expressions for Π together,

$$2\Pi = (abv)(a_s b_t - a_t b_s) M_1 .$$

But, by identity VII,[*]

$$a_s b_t - a_t b_s = (abw), \quad w = (st),$$

—the elements w are cogredient with u and hence the notation is justified—and it appears that

$$\Pi = \tfrac{1}{2}(abv)(abw) M_1 = v_\alpha w_\alpha \cdot \tfrac{1}{2} M_1 . \qquad \text{Q. E. D.}$$

We will, therefore, in the following always, when possible, replace symbols $a, b, \ldots.$ by symbols $\alpha, \beta, \ldots.$ The above applies equally well to the replacement of symbols $a', b', \ldots.$ by symbols $\alpha', \beta', \ldots.$

A corresponding theorem holds for products in which the symbols α, β appear in the combination $(\alpha\beta)_i$, $i = 1, 2, 3$. For, if the symbolic product Π contains the factor $(\alpha\beta s)$, then Π either contains a second factor $(\alpha\beta t)$ or else is of the form:

$$\Pi = (\alpha\beta s) M = (\alpha\beta s) v_\alpha w_\beta M',$$

where M' is free from α, β; and then

$$2\Pi = (\alpha\beta s)(v_\alpha w_\beta - v_\beta w_\alpha) M_1 .$$

By identity VIII, $\quad v_\alpha w_\beta - v_\beta w_\alpha = (\alpha\beta t), \quad t = (vw),$

and Π is also in this case of the form:

$$\Pi = (\alpha\beta s)(\alpha\beta t) \cdot \tfrac{1}{2} M_1 .$$

All rational integral invariants of f and f' are formed of terms that are products of symbolic factors of the types[†]

$$(abc), \quad (aba'), \quad (aa'b'), \quad (a'b'c'), \quad (abu), \quad (aa'u), \quad (a'b'u), \quad a_x, \quad a'_x .$$

Introducing the symbols α, α', we must annex to the foregoing further the symbolic factors $(\alpha\beta x)$, $(\alpha'\beta' x)$, $(a\alpha' x)$, and hence we see that *every rational integral invariant of f and f′ is a sum of products of factors of the following types:*

$$\left.\begin{matrix} a_x, & a'_x, & u_\alpha, & u_{\alpha'}, \\ a_\alpha, & a'_\alpha, & a_{\alpha'}, & a'_{\alpha'}, \\ (a\alpha' u), & (\alpha\beta x) & (\alpha'\beta' x), & (a\alpha' x), \end{matrix}\right\}, \tag{3}$$

each product multiplied by a numerical constant.

---

[*] The identities referred to in this paper are those given in *Symb. Not.*, §6.

[†] Cf. *Symb. Not.*, §4.

§2.—*The Method of Solution of the Problem of the Present Paper.*

Before attempting to describe the general method of solution of the problem of this paper, I will illustrate the method by a few simple examples.

a). A symbolic product Π of factors of the types (3) that contains the symbolic factor a_α contains the actual factor $a_\alpha^2 = A_{111}$. For Π is either of the form

$$\Pi = (abc)a_r b_s c_t \cdot M \quad \text{or} \quad = (abc)(abv)c_t \cdot M,$$

where M does not contain the symbols a, b, c, and r, s, t and v have such values as in the previous paragraph. By identity VI,

$$a_r b_s c_t + a_t b_r c_s + a_s b_t c_r - a_r b_t c_s - a_s b_r c_t - a_t b_s c_r = (abc)(rst).$$

Multiplying each side of this identity by $(abc) \cdot M$, we see that each of the six expressions on the left hand side represents Π, and hence

$$6\Pi = (abc)^2(rst)M, \quad \Pi = A_{111} \cdot \tfrac{1}{6}(rst)M.$$

In the second case, by identity IV,

$$(abv)c_t - (cbv)a_t - (acv)b_t = (abc)v_t.$$

Multiplying by $(abc)M$, we have

$$3\Pi = (abc)^2 v_t M, \quad \Pi = A_{111} \cdot \tfrac{1}{3} v_t M.$$

Similarly for products containing the symbolic factor $a'_{\alpha'}$: such products contain the actual factor A_{222}.

b). Again, if a symbolic product Π contains a factor of the type $(\alpha\beta s)$ and hence (§1) also the symbolic factor $(\alpha\beta t)$:

$$\Pi = (\alpha\beta s)(\alpha\beta t)M,$$

then Π contains the factor A_{111}. (If the constituents of Π were all factors of the types (3), s could be only x; but it is often convenient, as will appear later, to consider products in which $s = (au)$, $(a'u)$, (aa'), etc.) For,

$$(\alpha\beta s) = \begin{vmatrix} (ab)_1 & (ab)_2 & (ab)_3 \\ (cd)_1 & (cd)_2 & (cd)_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = (abd)c_s - (abc)d_s,$$

$$(\alpha\beta s)^2 = 2(abc)^2 d_s^2 - 2(abc)(abd)c_s d_s,$$

since $(abd)^2 c_s^2 = (abc)^2 d_s^2$. But the second term can be reduced to simpler form, for, by interchanging equivalent symbols, we find that

$$(abc)d_s\,.\,(abd)c_s = \tfrac{1}{3}(abc)d_s\{(abd)c_s - (cbd)a_s - (acd)b_s\} = \tfrac{1}{3}(abc)^2 d_s^2$$

by identity IV, and hence

$$(\alpha\beta s)^2 = \tfrac{4}{3}(abc)^2 d_s^2 = \tfrac{4}{3}A_{111}a_s^2.$$

The polar operator, $\frac{1}{2}\left(t_1\frac{\partial}{\partial s_1} + t_2\frac{\partial}{\partial s_2} + t_3\frac{\partial}{\partial s_3}\right)$, applied to each side of this equation, gives

$$(\alpha\beta s)(\alpha\beta t) = \tfrac{4}{3}A_{111}a_s a_t.$$

And similarly,

$$(\alpha'\beta' s)(\alpha'\beta' t) = \tfrac{4}{3}A_{222}a'_s a'_t.$$

Thus it appears that every symbolic product, Π, that contains any one of the symbolic factors a_α, $a'_{\alpha'}$, $(\alpha\beta s)$, $(\alpha'\beta' s)$, breaks up into two actual factors, one of which (A_{111}, A_{222}) belongs to those invariants (2) that we have already admitted to the system, and the other of which (M) contains fewer symbols than Π; and hence a symbolic product will surely be reducible to simpler products if it contains other factors than those of the following types:

$$\left.\begin{array}{cccc} a_x, & a'_x, & u_\alpha, & u_{\alpha'}, \\ & a_{\alpha'}, & a'_\alpha, & \\ & (aa'u), & (aa'x). & \end{array}\right\} \qquad (4)$$

The method of establishing the system of f and f' consists now in ascertaining still further restrictions on the choice of symbols for a symbolic product that shall really be an irreducible invariant, and not a product of two invariants, one (or both) of which contains fewer symbols than Π, or a sum of such products. It will appear that the number of such irreducible products is very limited, and these, taken with (1) and (2), form a set of invariants, in terms of which every symbolic product, and hence every simultaneous invariant of f and f', can be expressed rationally and integrally, *i. e.* they form the system of f and f'.

§3.—*Further Reduction of Symbolic Products.*

I will now show that a symbolic product containing two or more cogredient symbols of f: a, b, ... *or* α, β, ..., *i. e.* a product Π of the form

$$\Pi = a_r a_s b_t b_z M \quad \text{or} \quad = \rho_\alpha \sigma_\alpha \tau_\beta \omega_\beta M,$$

is expressible in terms of symbolic products each containing fewer symbols than Π. The same proof applies to products containing two or more cogredient symbols of f': a', b', ... *or* α', β', ... Hence it follows that all symbolic products can be expressed in terms of such as contain at most one of each of the symbols a, α, a', α'.

Theorem A.—In the symbolic products,

$$\Pi = a_r a_s b_t b_z M, \tag{I}$$

$$\Pi = \rho_\alpha \sigma_\alpha \tau_\beta \omega_\beta M, \tag{II}$$

either two of the elements r, s, t, z *or* ρ, σ, τ, ω *are identical or else* Π *can be expressed in terms of products for which this is the case, and of products that contain fewer symbols than* Π. *And similarly for products containing two or more symbols of* f'.

First consider the product (I). From the last § it is evident that we may restrict our attention to those cases in which r, s, t, z are four of the elements (see scheme (4)):

$$r,\ s,\ t,\ z: \begin{cases} x, \\ \alpha',\ \beta',\ \gamma',\ \ldots. \\ (a'u),\ (b'u),\ \ldots. \end{cases} \tag{5}$$

In no case can more than one of the elements r, s, t, z belong to the last row of (5); for otherwise Π would contain one of the two factors:

$$a_r a_s b_t b_z = \begin{cases} (aa'u)(ab'u)b_t b_z \\ (aa'u)(bb'u)a_t b_z, \end{cases}$$

where t, z are any two of the elements of (4). In the first case Π would be of the form:

$$\Pi = a'_{r'} a'_{s'} b'_{t'} b'_{z'} M, \quad \text{where} \quad r' = t' = (au),$$

and would, therefore, satisfy the conditions of the theorem, enunciated for the symbols of f'. In the second case, the application of identity IV:

$$(bb'u)a_t = (b'ua)b_t - (uab)b'_t + (abb')u_t$$

gives

$$(aa'u)(bb'u)a_t b_z = (aa'u)(ab'u)b_t b_z - (aa'u)(abu)b_z b'_t + (aa'u)(abb')b_z u_t.$$

The first of these symbolic products is of the form just considered; in the second and third, a, b can be replaced by the single symbol α and the symbolic products arising from these terms will contain fewer symbols than Π. Hence the theorem is proved for all products, Π, in which two or more of the symbols r, s, t, z belong to the last row of (5).

After the exclusion of such products, there remain to be considered products of the following three types:

$$a_r a_s b_t b_z = \begin{cases} a_{\alpha'} a_{\beta'} b_{\gamma'} b_z \\ a_{\alpha'} a_{\beta'} b_x b_z \\ a_{\alpha'} a_z \, b_x b_{\beta'} \end{cases}$$

where z is arbitrary. The first and second of these products already satisfy the conditions of the theorem, for Π is then of the form:

$$\Pi = \rho_{\alpha'} \sigma_{\alpha'} \tau_{\beta'} \omega_{\beta'} M, \quad \text{where} \quad \rho = \tau = a.$$

The third is also of a form already considered, unless $z = (a'u)$. Then

$$\Pi = (aa'u) a_{\alpha'} b_\beta b_x \,.\, \rho_{\alpha'} \sigma_{\beta'} M,$$

where ρ, σ are of the following types: u, c, d, $[c', d',]$ (αx), (βx), $[(\gamma' x), (\delta' x)]$. For the bracketed symbols Π would be directly reducible (§§1 and 2). If $\rho = (\alpha x)$, then Π contains the factor (identity VI):

$$(aa'u)(\alpha\alpha' x) = \begin{vmatrix} a_\alpha & a'_\alpha & u_\alpha \\ a_{\alpha'} & a'_{\alpha'} & u_{\alpha'} \\ a_x & a'_x & u_x \end{vmatrix} \tag{6}$$

and is, by the foregoing, reducible. Hence ρ, σ can be only of the form u, c, d. One of them, ρ, let us say, must, therefore, $= c$, and Π contains the factor $a_s a_{\alpha'} c_{\alpha'} c_t$, where $s = (a'u)$ and t may have any value. This product satisfies the conditions of the theorem, and thus the proof of the theorem for products of the form (I) is complete.

It remains to consider products of the form (II). This is the dualistic of the case just considered and the treatment is accordingly analogous. The reductions differ, however, in certain details from the previous ones, and for this reason I give them at length.

Only products in which ρ, σ, τ, ω are four of the elements:

$$\rho, \sigma, \tau, \omega : \begin{cases} u, \\ a', b', c' \ldots . \\ (\alpha' x), (\beta' x), \ldots . \end{cases} \tag{7}$$

need be considered. Not more than one element, ρ, can belong to the last row, for otherwise we should have

$$\rho_\alpha \sigma_\alpha \tau_\beta \omega_\beta = \begin{cases} (\alpha\alpha' x)(\alpha\beta' x) \tau_\beta \omega_\beta \\ (\alpha\alpha' x)(\beta\beta' x) \tau_\alpha \omega_\beta . \end{cases}$$

In the first case, Π has the factor $\rho'_{\alpha}\rho'_{\beta'}$ and thus satisfies the conditions of the theorem; in the second case (identity IV)*

$$(\alpha\alpha' x)\omega_{\beta} \,.\, (\beta\beta' x)\tau_{\alpha} = (\alpha\alpha' x)\omega_{\beta}\{(\beta' x\alpha)\tau_{\beta} - (x\alpha\beta)\tau_{\beta'} + (\alpha\beta\beta')\tau_{x}\}.$$

The first term is of the form just considered; the second and third have the factor $(\alpha\beta x)$ or $(\alpha\beta\beta')$, and these latter terms contain accordingly (§§1, 2) the actual factor A_{111}; they are, therefore, expressible in terms of products that contain fewer symbols than Π.

Lastly, when at least three of the elements ρ, σ, τ, ω belong to the first two rows of (7),

$$\rho_{\alpha}\sigma_{\alpha}\tau_{\beta}\omega_{\beta} = \begin{cases} a'_{\alpha}b'_{\alpha}\,c'_{\beta}\omega_{\beta} \\ a'_{\alpha}b'_{\alpha}u_{\beta}\omega_{\beta} \\ a'_{\alpha}\omega_{\alpha}u_{\beta}b'_{\beta} \end{cases}$$

where ω is arbitrary. The first and second of these products are already of the form demanded by the theorem, the third needs to be considered only when $\omega = (\alpha' x)$. Π then contains the factor

$$(\alpha\alpha' x)a'_{\alpha}b'_{\beta}u_{\beta} \,.\, a'_{r}b'_{s},$$

where r, s are of the following types: x, γ, δ, $[\gamma'$, $\delta',]$ (au), (bu), $[(c'u)$, $(d'u)]$, and it is evident from similar considerations to those of the previous case that the theorem holds for this product.

The proof of the theorem for products in the symbols of f is thus complete. Interchanging primed and unprimed symbols, we obtain at once the proof for the corresponding products in the symbols of f'.

Theorem B.—Symbolic products

$$\Pi = a_{r}a_{s}b_{t}b_{z}M, \qquad \text{(I)}$$
$$\Pi = \rho_{\alpha}\sigma_{\alpha}\tau_{\beta}\omega_{\beta}M, \qquad \text{(II)}$$

in which two of the symbols r, s, t, z *or* ρ, σ, τ, ω *are identical, are expressible in terms of symbolic products containing fewer symbols than* Π. *The same also holds for similar products in the symbols of* f'.

---

* It is to be noticed that symbolic determinant factors have the invariant property only when all the symbols are cogredient. Hence when a, b, c of identity IV are cogredient with x, the identity must be written in the form

$$(rst)d_{x} - (stx)d_{r} + (txr)d_{s} - (xrs)d_{t} \equiv 0\,.$$

For products of the form (I) there are three cases to consider:

$$\begin{array}{ll} 1)\ \ r = t = x & \left\{\begin{array}{ll} \text{a)} & s = \alpha' \\ \text{b)} & s = (\alpha' u) \end{array}\right. \\ 2)\ \ r = t = \alpha' & \\ 3)\ \ r = t = (\alpha' u) & \end{array}$$

1) a) $$a_x a_{\alpha'} b_x b_s = a_x b_s\left(a_x b_{\alpha'} + (a\alpha' x)\right)$$

b) $$(a\alpha' u)a_x b_x b_s = a_x b_s\{(\alpha' u b)a_x - (uba)\alpha'_x + (ba\alpha')u_x\}$$

2) $$a_{\alpha'} a_s b_{\alpha'} b_s = a_{\alpha'} b_s\{a_{\alpha'} b_s + (as\alpha')\}$$

3) $$(a\alpha' u)(b\alpha' u)a_s b_s = (b\alpha' u)a_s\{(\alpha' u b)a_s - (uba)\alpha'_x + (ba\alpha')u_x\}$$

by the help of identities IV and VII. The terms on the right hand side either contain as-factors terms with a less number of symbols than II or admit the replacement of the symbols a, b by the single symbol a, and hence meet in every case the demands of the theorem.

It is not necessary to make these divisions into cases, for the reductions for all four cases are comprised in the single formula:

$$a_r a_s b_r b_s = a_r b_s\{a_r b_s + (asr)\}. \tag{8}$$

Products of the form (II) correspond dualistically to those just considered and are treated in a similar manner. As before, three cases present themselves:

$$\begin{array}{ll} 1)\ \ \rho = \tau = u & \left\{\begin{array}{ll} \text{a)} & \sigma = \alpha' \\ \text{b)} & \sigma = (\alpha' x) \end{array}\right. \\ 2)\ \ \rho = \tau = \alpha' & \\ 3)\ \ \rho = \tau = (\alpha' x). & \end{array}$$

The desired reduction in each case is given by the formula:

$$\rho_\alpha \sigma_\alpha \rho_\beta \tau_\beta = \rho_\alpha \tau_\beta\{\sigma_\beta \rho_\alpha + (\alpha\beta t)\}, \quad \text{where} \quad t = (\sigma\rho), \tag{9}$$

and the theorem is proved.

By the repeated application of these two theorems, it is evident that every symbolic product can be ultimately expressed in terms of symbolic products, no one of which contains two cogredient symbols of f or two cogredient symbols of f'.

§4.—*The System of f and f'.*

It is now an easy task to write down a set of invariants in terms of which every other rational integral simultaneous invariant of f and f' is expressible rationally and integrally. A set of symbolic products, each of which contains at most *one* symbol of each of the four types $a, b \ldots\ldots, \alpha, \beta \ldots\ldots, a', b' \ldots\ldots, \alpha', \beta' \ldots\ldots$ is such a set, and is obtained, with the exception of two of its members, $A_{111} = a_\alpha^2$ and $A_{222} = a'^2_{\alpha'}$, by combining the symbolic factors (4) with each other in all possible ways. But the set thus obtained may be restricted still further, for from identity VI (see (6)) it is clear that any such product having the factor $(aa'u)(\alpha\alpha'x)$ can be replaced by products that do not contain these two determinant factors, and hence such products may be excluded from the set. The symbolic products that now remain are the following:

$$\left.\begin{array}{ll}
f = a_x^2 \quad , & f' = a'^2_x \quad , \\
F_{11} = u_\alpha^2 \quad , & F_{22} = u_{\alpha'}^2 \quad , \\
A_{111} = a_\alpha^2 \quad , & A_{222} = a'^2_{\alpha'} \quad , \\
A_{112} = a'^2_\alpha \quad , & A_{122} = a^2_{\alpha'} \quad , \\
B_1 = a'_\alpha a'_x u_\alpha \quad , & B_2 = a_{\alpha'} a_x u_{\alpha'} \quad , \\
 & \\
N = (aa'u) a_x a'_x \quad , & \mathrm{N} = (\alpha\alpha'x) u_\alpha u_{\alpha'} \quad , \\
C_1 = (aa'u) a'_\alpha a_x u_\alpha \quad , & \Gamma_1 = (\alpha\alpha'x) a_{\alpha'} u_\alpha a_x \quad ; \\
C_2 = (aa'u) a_{\alpha'} a'_x u_{\alpha'} \quad , & \Gamma_2 = (\alpha\alpha'x) a'_\alpha u_{\alpha'} a'_x \quad , \\
D = (aa'u) a_{\alpha'} a'_\alpha u_\alpha u_{\alpha'} , & \Delta = (\alpha\alpha'x) a_{\alpha'} a'_\alpha a'_x a_x , \\
 & \\
F_{12} = (aa'u)^2 \quad , & \Phi_{12} = (\alpha\alpha'x)^2 \quad .
\end{array}\right\} \quad \begin{array}{c}\text{The System of} \\ f \text{ and } f'.\end{array}$$

No one of these twenty invariants is expressible rationally and integrally in terms of the others and of the identical invariant u_x. *These invariants form the system of f and f'.*

Between these invariants exist certain identical relations of higher order.* Thus it is easy to show† that

$$u_x D \equiv F_{11} C_2 + F_{22} C_1 - \mathrm{N} F_{12}$$

and, dualistically,

$$u_x \Delta \equiv f \Gamma_2 \quad + f' \Gamma_1 \quad - N \Phi_{12} .$$

For, the first of these identities is, symbolically written,

$$(aa'u) a_{\alpha'} a'_\alpha u_\alpha u_{\alpha'} u_x \equiv (aa'u) a_{\alpha'} a'_x u_\alpha^2 u_{\alpha'} + (aa'u) a'_\alpha a_x u_{\alpha'}^2 u_\alpha - (aa'u)^2 (\alpha\alpha'x) u_\alpha u_{\alpha'} ,$$

or

$$(aa'u) u_\alpha u_{\alpha'} \{ (aa'u)(\alpha\alpha'x) + a_{\alpha'} a'_\alpha u_x - a_{\alpha'} a'_x u_\alpha - a'_\alpha a_x u_{\alpha'} \} \equiv 0 .$$

---

* Gordan: *Über Büschel von Kegelschnitten*, Math. Ann., Bd. 19.

† Forsyth: *Systems of Ternariants that are Algebraically Complete*, Amer. Jour., Vol. 12, p. 58.

Now, by identity VI,

$$(aa'u)(\alpha\alpha'x) - a_\alpha a'_{\alpha'} u_x - a_x a'_\alpha u_{\alpha'} - a_{\alpha'} a'_x u_\alpha + a_\alpha a'_x u_{\alpha'} + a_{\alpha'} a'_\alpha u_x + a_x a'_{\alpha'} u_\alpha \equiv 0,$$

and it is sufficient to show that

$$(aa'u) u_\alpha u_{\alpha'} \{a_\alpha a'_{\alpha'} u_x - a_\alpha a'_x u_{\alpha'} - a_x a'_{\alpha'} u_\alpha\} \equiv 0.$$

This appears as follows. The first term has the symbolic factors a_α and $a'_{\alpha'}$, and is, therefore, (§2) of the form $A_{111} A_{222} u_x \,.\, M$. But M must vanish identically, for, as an invariant, it is a product of factors of the types $(\rho\sigma\tau)$, $(\rho\sigma u)$, u_x given in *Symb. Not.* §4, and out of the three u's that remain no such factors can be formed that do not vanish identically. And similarly for the other two terms; they contain respectively the factors A_{111} and A_{222}, and after these factors have been extracted, there are not enough elements left from which to form non-vanishing factors of the degree required. The above is, therefore, a true identity.

An extended discussion of the geometric signification of the invariants of the system for the conics f and f' is to be found in Clebsch-Lindemann, *Geometrie*, I, pp. 291–304; further simultaneous invariants of f and f' are there treated and expressed in terms of the invariants of the system.

CAMBRIDGE, MASS., *March*, 1892.

On Generating Systems of Ternary and Quaternary Linear Transformations.

By Henry S. White.

The following special cases of a general algebraic theorem admit in geometric form a proof so simple that the process which it postulates can be entirely visualized. Though we tacitly assume the reality of all points spoken of, this restriction is of no moment; for two points determine a line, three points a plane, whether the points be real or imaginary. The theorem now to be examined is fundamental in the Algebra of Linear Transformations. I shall exhibit proofs of the theorem only as particularized in the theories of ternary and quaternary forms, since beyond three dimensions our intuition of spacial figures ceases. A general proof will necessarily be of purely algebraic character.

In order to write as simply as possible the differential equations satisfied by every invariant of linear transformation of n homogeneous variables, one seeks a system of linear transformations of simplest type, such that by repetition and successive application of these elementary transformations the most general can be compounded. Such a system is termed a complete system of Generators of the n-ary group.

§1.—*Generators of the Ternary Group.*

Theorem: A complete generating system of ternary linear transformations is contained in the following five; three of which have the determinant equal to 1, *the other two a determinant different from* 1.

$$\text{I.}\quad x_1 = y_1 + \lambda . y_2,\quad x_2 = y_2,\quad x_3 = y_3.$$

$$\text{II.}\quad x_1 = y_1,\quad x_2 = y_2 + \mu . y_3,\quad x_3 = y_3.$$

$$\text{III.}\quad x_1 = y_1,\quad x_2 = y_2,\quad x_3 = y_3 + \nu . y_1.$$

IV. $x_1 = y_1,$
$x_2 = \alpha . y_2,$
$x_3 = y_3.$

V. $x_1 = y_1,$
$x_2 = y_2,$
$x_3 = \beta . y_3.$

Interpreting $x_1 : x_2 : x_3$ and $y_1 : y_2 : y_3$ as point coordinates in a plane, I shall first give a geometric characterization of each of these five elementary transformations. The well-known method of determining completely by the use of two point-quadruples a general linear transformation will enable me, secondly, to restate in geometric terms the above theorem in the form of a problem. A simple process of solution suggests itself, whose verification finally proves the theorem.

By the substitutions IV and V each side and each vertex of the triangle of reference in the x-plane becomes the corresponding side or vertex of the triangle of reference in the y-plane. Calling the vertices A_1, A_2, A_3, the opposite sides a_1, a_2, a_3, I may say more briefly: Each of the six elements A_1, A_2, A_3, a_1, a_2, a_3 of the triangle of reference is transformed into itself. So is also, by IV, every line $ax_1 + cx_3 = 0$ passing through A_2, and by V, every line $ax_1 + bx_2 = 0$ passing through A_3. But every point not on the stationary line ($x_2 = 0$ in the one case, $x_3 = 0$ in the other) is transferred directly toward or directly from the fixed point A_2 or A_3. Hence I may call these transformations IV and V, loosely, *translations* of the x-plane.

Similarly the transformations I, II, III may be called for present purposes *rotations* of the plane. By the transformation I, for example, while the sides $x_2 = 0$ and $x_3 = 0$ become the sides $y_2 = 0$ and $y_3 = 0$ of the new triangle of reference, the side $y_1 = 0$ coincides, not with $x_1 = 0$, but with the line $x_1 - \lambda . x_2 = 0$. Every line through the vertex A_1 $ax_1 + bx_2 = 0$ becomes a line through the new vertex A'_1 $ay_1 + (a\lambda + b)y_2 = 0$. The transformation is described, with sufficient precision, as a *rotation of the side a_1 about the vertex A_3 as a center.* Fixing the attention upon the triangle of reference I see that—

By I, the side a_1 revolves about A_3 as a center.
" II, " " a_2 " " A_1 " " "
" III, " " a_3 " " A_2 " " "

The theorem gives as elementary operations, therefore, three rotations and two translations. By the aid of these special transformations there is to be produced the total effect of any given general collineation. What is the geometrical character of the latter?

Projective Geometry furnishes the theorem: "A collineation is completely determined when to any four points of the plane are assigned respectively the four points into which they are to be transformed; provided that no three points of either quadruple lie in a straight line." Taking as given points A_1, A_2, A_3, and P not lying on either a_1, a_2, or a_3, and assigning to them respectively four arbitrary points B_1, B_2, B_3, Q, I shall have proved the required theorem if I can always solve the following problem:

By the use of the three rotations and two translations above described, to transfer the three vertices A_1, A_2, A_3 of the triangle of reference, and a fourth point P not lying upon any side of that triangle, to four arbitrary points B_1, B_2, B_3, Q respectively, where no three of the latter four points are collinear.

The solution is effected as follows. If the three vertices A_1, A_2, A_3 were already brought to occupy the positions B_1, B_2, B_3, they could remain fixed while the two translations should be applied to transfer the fourth point, P, to the position Q. Now the three rotations suffice for transferring the vertices A_1, A_2, A_3 to the desired points; for by their aid it is possible to transfer any one, *e. g.* A_1, to an arbitrary point B_1 and leave the other two, A_2 and A_3, at their initial positions. To see the truth of this assertion, suppose that B_1 is taken upon the a_3. A rotation III withdraws the side a_3 from the point B_1, and transfers the vertex A_1 along the side a_2 for an arbitrary distance to some point A'_1. If B_1 were taken not lying upon a_3, A_1 may be said to coincide with A'_1. Next

By II, let a_2 be brought to contain B_1; incidentally the vertex A_3 moves upon a_1 to a position A'_3;

By III, let a_3 be brought to contain B_1; this completes the transfer of A'_1 to B_1; finally—

By II, let a_2 be revolved about B_1 till its intersection with a_1 returns from A'_3 to its initial position A_3.

Thus by at most four rotations, III, II, III, II, A_1 has been transferred to B_1, while A_2 and A_3 are unchanged. If P has been brought to P', then next in order—

By I, III, I, III, the points B_1, A_2, A_3, P' may become B_1, B_2, A_3, P'';
" II, I, II, I, " " B_1, B_2, A_3, P'' " " B_1, B_2, B_3, P'''.

If now the lines $\overline{B_2P''}$ and $\overline{B_3Q}$ intersect, or if the lines $\overline{B_2Q}$ and $\overline{B_3P'''}$ are not parallel, two translations will complete the required transfer; otherwise three will be necessary. Neglecting the latter possibility, since it adds no difficulty to the problem, suppose $\overline{B_2P''}$ and $\overline{B_3Q}$ to intersect in a point P'''.

By IV, let P'' be brought to the position P''';
" V, " P''' " " " Q.

These translations leave B_1, B_2, B_3 fixed, and bring the point whose original position was P to the terminal position Q. The above problem is therefore solved, and the theorem stated at the outset is proved.

Logically more symmetrical, but practically less brief would be the solution, if made to depend on the lemma: *By using not more than seven operations chosen from the three rotations and two translations, it is possible to transfer any one of the four points* A_1, A_2, A_3, P, *to the corresponding point* B_1, B_2, B_3 *or* Q *in such a way that the terminal positions of the other three shall coincide with their initial positions.* The proof is sufficiently obvious from the foregoing. An interesting exceptional case would arise, for example, if the points B_1, B_2, B_3 were situated upon the sides a_1, a_2, a_3 respectively, and the fourth point Q lay upon any one side a_1, a_2, or a_3 of the triangle of reference.

§2.—*Generators of the Quaternary Group.*

For the group of linear transformations of four homogeneous coordinates $x_1 : x_2 : x_3 : x_4$ in three-dimensional space an obvious extension of the theorem of §1 is the following:

Theorem: A complete generating system of quaternary linear transformations is contained in the following seven, each of which contains a single variable parameter.

$$
\begin{array}{llcll}
\text{I.} & x_1 = y_1 + \lambda_1 y_2, & \qquad & \text{II.} & x_1 = y_1, \\
 & x_2 = y_2, & & & x_2 = y_2 + \lambda_2 y_3, \\
 & x_3 = y_3, & & & x_3 = y_3, \\
 & x_4 = y_4. & & & x_3 = y_4.
\end{array}
$$

$$\text{III.}\quad \begin{aligned} x_1 &= y_1, \\ x_2 &= y_2, \\ x_3 &= y_3 + \lambda_3 y_4, \\ x_4 &= y_4. \end{aligned} \qquad\qquad \text{IV.}\quad \begin{aligned} x_1 &= y_1, \\ x_2 &= y_2, \\ x_3 &= y_3, \\ x_4 &= y_4 + \lambda_4 y_1. \end{aligned}$$

$$\text{V.}\quad \begin{aligned} x_1 &= y_1, \\ x_2 &= \alpha \,.\, y_2, \\ x_3 &= y_3, \\ x_4 &= y_4. \end{aligned} \qquad \text{VI.}\quad \begin{aligned} x_1 &= y_1, \\ x_2 &= y_2, \\ x_3 &= \beta \,.\, y_3, \\ x_4 &= y_4. \end{aligned} \qquad \text{VII.}\quad \begin{aligned} x_1 &= y_1, \\ x_2 &= y_2, \\ x_3 &= y_3, \\ x_4 &= \gamma \,.\, y_4. \end{aligned}$$

The order and method of proof employed in the preceding case can be followed again here. Any particular value of the arbitrary parameter in each of these seven elementary transformations shall be regarded as the terminal value of a quantity which has varied continuously from an initial value zero. Each such finite variation of a parameter can be represented by a system of motions of finite magnitude in three-dimensional space. In the tetraedron of reference let each face: $x_\iota = 0$ be named the face a_ι; let each vertex opposite a face a_ι be called A_ι, and each edge $\overline{A_\iota A_\kappa}$ be called $\alpha_{\iota\kappa}$. The above seven transformations may then be termed respectively *axial rotations* (I, II, III, IV) or *translations* (V, VI, VII). The effects which are relevant to the present purpose may be noted briefly, as follows: Each axial rotation leaves *in situ* three faces, three vertices, and four edges, of the tetraedron of reference; but causes to revolve about one edge as an axis every plane, save one, containing that edge,—among others the fourth face, containing the fourth vertex, of the tetraedron of reference. More particularly—

Rotation I, about an axis α_{34}, shifts the face a_1 and vertex A_2,
" II, " " " α_{41}, " " " a_2 " " A_3,
" III, " " " α_{12}, " " " a_3 " " A_4,
" IV, " " " α_{23}, " " " a_4 " " A_1.

Each translation leaves *in situ* every vertex of the tetraedron of reference, but transfers directly toward or from some one vertex every point not lying in the opposite face of the tetraedron. For the translations V, VI, VII, the central vertices are respectively A_2, A_3, A_4.

Since a collineation is completely determined when it is required to transform five given points into arbitrary points respectively, no four points in either quintuple being co-planar, the proof of the above theorem will be contained in the solution of the problem:

By means of operations chosen from the four sorts of rotation, and the three sorts of translation just described, to transfer the four vertices A_1, A_2, A_3, A_4 *of the tetraedron of reference and a fifth point* P *not lying in any face thereof, to five arbitrary positions* B_1, B_2, B_3, B_4 and Q *respectively, no four of which lie in the same plane.*

The solution will correspond, step for step, to that adopted in the ternary case. Each point A_1, A_2, A_3, or A_4 in succession can be transferred by five, six, or seven *rotations* to the corresponding terminal position while the remaining three vertices remain undisturbed or else are restored to their original positions. If B_1, for example, is not in either face a_3 or a_4, then by the following series of rotations:

II, III, IV, III, II, the points
A_1, A_2, A_3, A_4, P can be brought to the
positions B_1, A_2, A_3, A_4, P'. Similarly three other sets of five or more rotations can be applied to transfer these five points to the positions:

$$B_1, B_2, B_3, B_4, P^{(IV)}.$$

It remains only to bring the point $P^{(IV)}$ to the position Q by means of translations. If either

the plane $(A_2, A_3, P^{(IV)})$ is not parallel to $\overline{A_4Q}$,
or " " $(A_3, A_4, P^{(IV)})$ " " " " $\overline{A_2Q}$,
" " " $(A_4, A_2, P^{(IV)})$ " " " " $\overline{A_3Q}$,

then three translations suffice to effect the object; otherwise, four. Thus we have, for the general case, a complete solution of the problem, which is tantamount to a proof of the theorem. In the most highly specialized case, four arbitrary rotations of the four sorts and any one arbitrary translation would reduce the problem to one of general character. Hence for all cases the theorem is established.

§3.—*Algebraical Determination of the Generators of a given Ternary Linear Transformation.*

When the nine coefficients of a ternary transformation:

$$\begin{aligned}\rho x_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3\\ \rho x_2 &= a_{21}y_1 = a_{22}y_2 + a_{23}y_3\\ \rho x_3 &= a_{31}y_1 = a_{32}y_2 + a_{33}y_3\end{aligned}$$

are given, and it is desired to find the numerical values of the parameters in its generators, the scheme of §1 gives a most convenient algorithm for the purpose. First we must determine the coordinates of the positions into which the four points—

$$\left\{\begin{array}{llll} A_1, & \text{whose coordinates are} & & 1:0:0 \\ A_2, & \text{''} \qquad \text{''} & \text{''} & 0:1:0 \\ A_3, & \text{''} \qquad \text{''} & \text{''} & 0:0:1 \\ P_1 & \text{''} \qquad \text{''} & \text{''} & p_1, p_2, p_3 \end{array}\right.$$

are to be transferred. These are readily seen. Calling the determinant of the substitution D, —

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ and its minors } D_{ik}, —$$

$D_{ik} = \dfrac{\partial D}{\partial a_{ik}}$, we have as coordinates $(b_{i1} : b_{i2} : b_{i3})$ of each point B_i, $(i = 1, 2, 3)$, the following:

$$\left.\begin{array}{l} \sigma . b_{i1} = D_{i1} \\ \sigma . b_{i2} = D_{i2} \\ \sigma . b_{i3} = D_{i3} \end{array}\right\} \begin{array}{l} (i = 1, 2, 3), \\ \sigma = \dfrac{D}{\rho}. \end{array}$$

P is transformed into Q with the coordinates:

$$\left\{\begin{array}{l} \sigma . q_1 = p_1 . D_{11} + p_2 . D_{21} + p_3 . D_{31} \\ \sigma . q_2 = p_1 . D_{12} + p_2 . D_{22} + p_3 . D_{32} \\ \sigma . q_3 = p_1 . D_{13} + p_2 . D_{23} + p_3 . D_{33} \end{array}\right\}.$$

The algebraic equivalent of transferring either one of the four points to the required terminal position is the determination of suitable values for two parameters. There are therefore in all eight parameters to be fixed—exactly the number of coefficients, barring the factor of proportionality, that occur in the given linear transformation.

For effecting the transfer of A_1 to B_1 while A_2 and A_3 remain *in situ*, the generators have the form,—

$$\left\{\begin{array}{l|l|l} x_1 = x_1' & x_1' = x_1'' & x_1'' = x_1''' \\ x_2 = x_2' + \mu_1 x_3' & x_2' = x_2'' & x_2'' = x_2''' - \mu_1 x_3''' \\ x_3 = x_3' & x_3' = x_3'' + \nu_1 x_1'' & x_3'' = x_3''' \end{array}\right.$$

The composition of these gives the substitution,—

$$\begin{cases} x_1 = x_1''' \\ x_2 = x_2''' + \mu_1\nu_1 x_1''' \\ x_3 = x_3''' + \nu_1 x_1''' \end{cases}$$

Inserting the coordinates of A_1 and B_1, we have to determine ν_1, μ_1, the formulæ:

$$\begin{cases} \sigma \,.\, 1 = D_{11} \\ \sigma \,.\, 0 = D_{12} + \mu_1\nu_1 D_{11} \\ \sigma \,.\, 0 = D_{13} + \nu_1 D_{11} \end{cases} \Bigg| \quad \therefore \begin{cases} \nu_1 = -\dfrac{D_{13}}{D_{11}} \\ \mu_1 = +\dfrac{D_{12}}{D_{13}} \end{cases}$$

Introducing these values of μ_1, ν_1, we find the coordinates of P',—

$$\begin{cases} p_1' = p_1 \\ p_2' = p_2 + \dfrac{D_{12}}{D_{11}} \cdot p_1 \\ p_3' = p_3 + \dfrac{D_{13}}{D_{11}} \cdot p_1 \end{cases}$$

In like manner, corresponding to the removal of A_2 and A_3 respectively, the two pairs of parameters ν_2, λ_2; λ_3, μ_3 would be found, and thence successively the coordinates of P'' and P'''. To transform the latter point into Q, there are to be determined finally the values of α, β from the formulæ (see IV and V, §1),—

$$\begin{cases} \sigma \,.\, p_1''' = \sigma \,.\, q_1 = p \,.\, D_{11} + p_2 \,.\, D_{21} + p_3 \,.\, D_{31} \\ \sigma \,.\, p_2''' = \alpha \,.\, \sigma \,.\, q_2 = \alpha \,.\, (p_1 \,.\, D_{12} + p_2 \,.\, D_{22} + p_3 \,.\, D_{32}) \\ \sigma \,.\, p_3''' = \beta \,.\, \sigma \,.\, q_3 = \beta \,.\, (p_1 \,.\, D_{13} + p_2 \,.\, D_{23} + p_3 \,.\, D_{33}) \end{cases}$$

Thus all eight parametric values are found from linear equations. The solution of these equations can become impossible only by the vanishing of one or more first minors of the determinant D. To the treatment of such exceptional cases the geometric considerations of §1 furnish still a perfect guide, requiring the insertion of additional generators, with arbitrary (restricted) parameters, in the system of eleven which suffices for the general case.

The analogous algorithm for factoring any given quaternary linear transformation into generators of the types here prescribed is sufficiently obvious, arising out of the scheme of §2 just as that given above for ternary forms arose from the scheme of §1. We see readily that it will fix the values of 15 parameters involved in 23 or more generators; and that if for a particular transformation more than 23 generators are requisite, all the parameters beyond 15 will be arbitrary. Further, it is worthy of remark that this algorithm, once found, is readily extended to determining similarly defined generators of an n-ary linear transformation. Hence there would be no difficulty in inferring that n *rotations* and $n-1$ *translations* must comprise a complete system of generators for the linear transformations of n homogeneous variables.

CLARK UNIVERSITY, WORCESTER, MASS., *March* 22, 1892.

A Symbolic Demonstration of Hilbert's Method for deriving Invariants and Covariants of given Ternary Forms.

By Henry S. White.

In the course of a recent paper, now fast becoming famous, Dr. Hilbert of Koenigsberg has developed a process of derivation by means of substitution which is likely to prove an important contribution to the theory of invariants and covariants.* Strictly speaking, the process itself is not new, but to Dr. Hilbert is due the credit of having adapted it to an entirely new end. Practically the same operation was employed by Clebsch,† Gordan,‡ and F. Mertens‖ in studying the formal constitution of invariants of linear transformation. Hilbert, detecting the rationale of this use, applies the operation to any term or sum of terms that can form part of an invariant expression, and shows that the result is a complete invariant. While the summary proof that he gives is sufficient and elegant, yet it seems possible to simplify both the statement of the operation and the proof of its character. For this purpose I employ the symbolic notation of Aronhold and Clebsch for the coefficients of a ternary form; thereby the invariant character of the result becomes evident upon mere inspection. Hilbert's method then admits of restatement as an aggregate of extremely simple groupings and substitutions; a statement which can then be extended at once, as Hilbert's has been, to quaternary and n-ary quantics.

The derivation of covariants is possible by the same method with slight modification, and the required covariant may contain as many cogredient sets of variables as one wishes. Further, it may be a function of several distinct sets of variables, each set subject to an independent linear transformation; *e. g.* it may be a combinant of several quantics of equal order. I shall give a general statement of the method as applied to ternary covariants, referring for proof to the discussion upon invariants. In conclusion I present, as a useful application, an easy proof of an important (and perhaps not sufficiently well-known) theorem on covariants.

---

* Ueber die Theorie der algebraischen Formen, Math. Ann. 36, pp. 524–6.

† Ueber symbolische Darstellung algebraischer Formen, Crelle's Journal, 59, pp. 7–15.

‡ Gordan-Kerschensteiner, Vorlesungen über Invariantentheorie, Bd. 2, pp. 114–119.

‖ Ueber invariante Gebilde ternärer Formen, Sitzungsberichte der Math.-Naturwissensch. Classe der kais. Akad. der Wissenschaften, Wien, 1887, pp. 942–992.

§1.—*Hilbert's Auxiliary Substitution and Cayley's Operator Ω combined and expressed in Clebsch-Aronhold Symbolic define a substitutional Operation giving rise to Invariants.*

In order to define the weights of a function of the coefficients of a ternary form, I follow a common notation in writing each coefficient with three subscript indices:

$$f(x_1, x_2, x_3) = \sum_{(i+k+l=n)} \frac{n!}{i!\,k!\,l!} \cdot a_{ikl} x_1^i x_2^k x_3^l. \tag{1}$$

These three indices may be termed the first, second, and third weights respectively of the coefficient a_{ikl}. In any product of coefficients the first weight will mean the sum of the first weights of the several factors, etc.,

$$A_{IKL} = a_{ikl} \cdot a_{i'k'l'} \cdot a_{i''k''l''} \ldots \text{(to } g \text{ factors)}. \tag{2}$$

$$\left.\begin{array}{ll} \text{First weight} & I = \Sigma(i) = i + i' + i'' + \ldots \text{(to } g \text{ terms)}, \\ \text{Second weight} & K = \Sigma(k), \\ \text{Third weight} & L = \Sigma(l) = n \,.\, g = (I + K). \end{array}\right\} \tag{3}$$

Every invariant of degree g in the coefficients of the ternary form $f(x_1, x_2, x_3)$ will be composed of terms all having the same first weight, the same second weight, and consequently the same third weight; and these three weights must be equal; *i. e.* if the invariant with its weights be designated by J_{IKL}, and its degree by g, we must have

$$I = K = L = \tfrac{1}{3} n \,.\, g. \tag{4}$$

The problem proposed is, *from a given sum of terms satisfying these conditions as to degree and weights, to derive by some determinate process a complete invariant.*

As a solution of this problem, the derivation-process adopted by Hilbert is the following, consisting of three distinct steps. Suppose the ternary form $f(x_1 x_2 x_3)$, (1), to have arisen from the form

$$\bar f(y_1, y_2, y_3) = \sum_{i+k+l=n} \frac{n!}{i!\,k!\,l!}\,\bar a_{ikl}\cdot y_1^i y_2^k y_3^l \equiv f(x_1, x_2, x_3) \tag{5}$$

by the linear substitution

$$\left\{\begin{array}{l} y_1 = \xi_1 x_1 + \eta_1 x_2 + \zeta_1 x_3 \\ y_2 = \xi_2 x_1 + \eta_2 x_2 + \zeta_2 x_3 \\ y_3 = \xi_3 x_1 + \eta_3 x_2 + \zeta_3 x_3 \end{array}\right\}. \tag{6}$$

Then the coefficients a_{ikl} of the form f are linear homogeneous functions of the coefficients $\bar a_{ikl}$ of $\bar f$, and contain moreover the parameters ξ, η, ζ homogeneously to the order n. Let the given polynomial satisfying the conditions (4) be denoted by

$$(A_{IKL}).$$

The first step is this: *Substitute in the polynomial (A_{IKL}) for the coefficients a_{ikl} their values expressed in terms of the coefficients $\bar a_{ikl}$ and the parameters, ξ's, η's, ζ's.* Call the resulting polynomial

$$(\bar A_{IKL}).$$

Its indices I, K, L now denote its orders in the ξ's, the η's, and the ζ's respectively. Remembering that each of these is equal to $\frac{1}{3}n\,.\,g$, (4), we come to the second step in the operation: *Free the expression from the nine parameters by applying $\frac{1}{3}n\,.\,g$ times in succession the operator Ω:*

$$\Omega = \begin{vmatrix} \dfrac{\partial}{\partial\xi_1} & \dfrac{\partial}{\partial\xi_2} & \dfrac{\partial}{\partial\xi_3} \\ \dfrac{\partial}{\partial\eta_1} & \dfrac{\partial}{\partial\eta_2} & \dfrac{\partial}{\partial\eta_3} \\ \dfrac{\partial}{\partial\zeta_1} & \dfrac{\partial}{\partial\zeta_2} & \dfrac{\partial}{\partial\zeta_3} \end{vmatrix}. \tag{7}$$

The resulting expression,

$$\Omega^{\frac{1}{3}ng}\,.\,(\bar A_{IKL}) = J(\bar a_{ikl}), \tag{8}$$

is a function of the coefficients $\bar a_{ikl}$. The third step is the following: *Replace in $J(\bar a_{ikl})$ every coefficient $\bar a_{i'k'l'}$ by the corresponding coefficient $a_{i'k'l'}$ of the form $f(x_1, x_2, x_3)$; the function $J(a_{ikl})$ is then an invariant of the latter form.*

The proof of the invariant property in this product of three operations I shall exhibit by analysis of the effect of successive steps, writing each symbolically. First of all as a necessary preliminary I suppose each actual coefficient, a_{ikl} or $\bar{a}_{ikl}$, replaced by a symbolic product $a_1^i a_2^k a_3^l (\equiv b_1^i b_2^k b_3^l \equiv \text{etc.})$ or $\bar{a}_1^i \bar{a}_2^k \bar{a}_3^l$; *i. e.* I write each form in the Clebsch-Aronhold notation, thus:

$$\left.\begin{array}{l} f(x_1, x_2, x_3) \equiv (a_1x_1 + a_2x_2 + a_3x_3)^n = a_x^n = b_x^n = c_x^n, \text{ etc.} \\ \bar{f}(y_1, y_2, y_3) \equiv (\bar{a}_1y_1 + \bar{a}_2y_2 + \bar{a}_3y_3)^n = \bar{a}_y^n = \bar{b}_y^n = \bar{c}_y^n, \text{ etc.} \end{array}\right\}. \tag{9}$$

By virtue of the linear relations (6) between the x's and the y's, the symbols a are expressed linearly in the symbols $\bar{a}$.*

$$\left\{\begin{array}{l} a_1 = \bar{a}_1\xi_1 + \bar{a}_2\xi_2 + \bar{a}_3\xi_3 \\ a_2 = \bar{a}_1\eta_1 + \bar{a}_2\eta_2 + \bar{a}_3\eta_3 \\ a_3 = \bar{a}_1\zeta_1 + \bar{a}_2\zeta_2 + \bar{a}_3\zeta_3 \end{array}\right\}. \tag{9a}$$

Hence follows, *e. g.* the expression of $f(x_1, x_2, x_3)$ in terms of symbols $\bar{a}$:

$$f(x_1, x_2, x_3) = (x_1\bar{a}_\xi + x_2\bar{a}_\eta + x_3\bar{a}_\zeta)^n.$$

From this one sees that the substitution of their values in terms of $\bar{a}_{ikl}$ and the parameters for the a_{ikl} in the polynomial (A_{IKL}) is effected by the operator $S^g_{\xi\eta\zeta}$:

$$(n\,!)^g \, . \, S^g_{\xi\eta\zeta} = \left\{\begin{array}{c} \left(\bar{a}_\xi \frac{\partial}{\partial a_1} + \bar{a}_\eta \frac{\partial}{\partial a_2} + \bar{a}_\zeta \frac{\partial}{\partial a_3}\right)^n . \\ \left(\bar{b}_\xi \frac{\partial}{\partial b_1} + \bar{b}_\eta \frac{\partial}{\partial b_2} + \bar{b}_\zeta \frac{\partial}{\partial b_3}\right)^n . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\text{to } g \text{ factors of } n^{\text{th}} \text{ order}). \end{array}\right\} \tag{10}$$

The second operator, Ω, is already symbolically expressed, (7). The third is simply this:

$$(n\,!)^g R^g(\bar{a}, a) = \left\{\begin{array}{c} \left(a_1 \frac{\partial}{\partial \bar{a}_1} + a_2 \frac{\partial}{\partial \bar{a}_2} + a_3 \frac{\partial}{\partial a_3}\right)^n . \\ \left(b_1 \frac{\partial}{\partial \bar{b}_1} + b_2 \frac{\partial}{\partial \bar{b}_2} + b_3 \frac{\partial}{\partial \bar{b}_3}\right)^n . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\text{to } g \text{ factors of } n^{\text{th}} \text{ order}). \end{array}\right\} \tag{11}$$

By the aid of these symbols we may write the final result thus:

$$J(a_{ikl}) = R^g(\bar{a}, a) \, . \, \Omega^{\frac{1}{3}ng} \, . \, S^g_{\xi\eta\zeta} \, . \, (A_{IKL}). \tag{12}$$

---

* Gordan, *loc. cit.*, pp. 110 and 114.

The advantage of this distinguishing of operators from function or *operand* is, that we are at liberty to combine $\Omega^{\frac{1}{3}ng}$ with $S^g_{\xi\eta\zeta}$ before applying. The simplest grouping of symbolic terms in the combined operation is to be sought; it appears to be the following. *Let the three constituents in any horizontal row of the three-rowed determinant* Ω *act simultaneously upon a single linear factor of* $S_{\xi\eta\zeta}$. The symbolic product $\Omega^{\frac{1}{3}ng} . S^g_{\xi\eta\zeta}$ is thereby expanded into a sum of terms, each of which is a product of $\frac{1}{3} . n . g$ determinants. The number of such terms is the number of permutations of the $n . g$ linear factors of $S^g_{\xi\eta\zeta}$. But every such term will vanish identically if the constituents of two rows in any one Ω have acted upon like factors of $S^g_{\xi\eta\zeta}$; *e. g.* upon the two factors:

$$\left(\bar{a}_\xi \frac{\partial}{\partial a_1} + \bar{a}_\eta \frac{\partial}{\partial a_2} + \bar{a}_\zeta \frac{\partial}{\partial a_3}\right)^2.$$

There will remain therefore only terms composed of symbolic factors of the following type:

$$\begin{vmatrix} \bar{a}_1 \dfrac{\partial}{\partial a_1} & \bar{a}_2 \dfrac{\partial}{\partial a_1} & \bar{a}_3 \dfrac{\partial}{\partial a_1} \\ \bar{b}_1 \dfrac{\partial}{\partial b_2} & \bar{b}_2 \dfrac{\partial}{\partial b_2} & \bar{b}_3 \dfrac{\partial}{\partial b_2} \\ \bar{c}_1 \dfrac{\partial}{\partial c_3} & \bar{c}_2 \dfrac{\partial}{\partial c_3} & \bar{c}_3 \dfrac{\partial}{\partial c_3} \end{vmatrix} \equiv (\bar{a}\bar{b}\bar{c}) \frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3}.$$

Indicating a term composed of $\frac{1}{3}ng$ such factors by $\prod_{(\frac{1}{3}ng)}\left[(\bar{a}\bar{b}\bar{c})\frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3}\right]$, we may express the combined operation thus:

$$\Omega^{\frac{1}{3}ng} . S^g_{\xi\eta\zeta} . (A_{IKL}) = \sum \prod_{(\frac{1}{3}ng)}\left[(\bar{a}\bar{b}\bar{c})\frac{\partial^3}{\partial a_1 . \partial b_2 . \partial c_3}\right](A_{IKL}), \tag{13}$$

where the summation extends over all possible arrangements of the symbols in each term of (A_{IKL}) in triplets such that each triplet consists of three different letters with three different indices (as $a_1 b_2 c_3$). Disregarding here the not uninteresting question as to the number of such possible arrangements, its maximum, minimum, etc., we have to notice only the formal constitution of the result. *Every term in the result is a product of* $\frac{1}{3}ng$ *factors of the type* $(\bar{a}\bar{b}\bar{c})$, *and is therefore per se invariant.* It remains finally to apply the operator $R^g(\bar{a}, a)$ whose effect is evidently *to replace each elementary determinant* $(\bar{a}\bar{b}\bar{c})$ *by* (abc), *so that the aggregate of terms becomes an invariant of* $f(x_1, x_2, x_3)$. While it would require arbitrary conventions to enable us to write the combination of the three operators in analytical symbols,—*e. g.*

$$\sum \prod_{(\frac{1}{3}ng)} \left[(abc)\right] \prod_{(\frac{1}{3}ng)} \left[\frac{\partial^3}{\partial a_1 \cdot \partial b_2 \cdot \partial c_3}\right],$$

yet it is easy to recapitulate in verbal form the entire process as analyzed above.

Hilbert's method for deriving an invariant of the groundform $f(x_1, x_2, x_3)$ *or* a_x^n, *from a homogeneous entire function having the degree g in the coefficients and having its three weights each equal to* $\frac{1}{3} . n . g$ *is equivalent to the following. Write each coefficient of the groundform as a symbolic product of n factors according to the Clebsch-Aronhold notation, using g different sets of n similar letters. Separate in all possible ways in every term of the function the* $n . g$ *factors into* $\frac{1}{3}ng$ *triplets, each triplet containing three different literal symbols and three subscript indices* 1, 2, 3. *Replace in each such arrangement every triplet* $a_1 b_2 c_3$ *by a corresponding three-rowed determinant* (abc). *The aggregate of resulting products is by virtue of its symbolic structure an invariant.*[*] *Q. e. d.*

This statement of the method is extended without difficulty to the derivation of an invariant from a function of the coefficients of several groundforms. Such a function is required to be homogeneous in the coefficients of each groundform separately; while in computing its three weights, which must be equal, subscript indices from all coefficients indiscriminately are to be summed up. The only formal feature in which the resulting invariant can differ from a product of invariants of single forms will be the occurrence of such symbolic factors as (aba'), $(aa'a'')$, etc.; where symbols a', a'' refer to additional groundforms $a_x'^{n_1}$, $a_x''^{n_2}$, different from the first, $a_x^n = b_x^n$. The additional groundforms may include one or more linear forms, whose coefficients we are free to regard as variables contragredient to the x_1, x_2, x_3. Simultaneous invariants involving those coefficients are usually termed *contravariants;* these demand no special treatment here. *Covariants* might be dismissed as equivalent to contravariants containing at least two sets of variables, but for convenience of application I shall state explicitly the process of deriving a covariant from a suitable function.

---

[*] A single, non-symbolic operation for effecting such derivation is given by Prof. W. E. Story in the Proceedings of the London Math. Soc. for 1891-2. The operator consists namely of a product of two series, infinite in form, but having only a finite number of terms applicable to any given case; each term denoting a particular succession of elementary substitutions.

§2.—*Modified Method for Covariants, with Corollary.*

By the weight of a covariant we usually understand the exponent of that power of the modulus, by which the covariant is multiplied when its variables are transformed linearly. To be consistent, we must define the weights of a single term in the covariant in the following way. Let B_{IKL}^{Θ} denote a term of degree g in the coefficients of a groundform a_x^n, and of order Θ in several sets of variables taken together:

$$\left.\begin{array}{l} B_{IKL}^{\Theta} = a_{ikl} \, . \, a_{i'k'l'} \, . \, a_{i''k''l''} \, . \, . \, . \, . \, x_1^{\alpha} x_2^{\beta} x_3^{\gamma} \, . \, x_1'^{\alpha'} x_2'^{\beta'} x_3'^{\gamma'} \, . \, . \, . \, . \\ \Sigma(i) + \Sigma(k) + \Sigma(l) = ng, \\ \Sigma(\alpha) + \Sigma(\beta) + \Sigma(\gamma) = \Theta. \end{array}\right\} \quad (14)$$

Let I, K, L denote the three weights; they must be equal to the expressions

$$\Sigma(i) - \Sigma(\alpha), \ \Sigma(k) - \Sigma(\beta), \ \Sigma(l) - \Sigma(\gamma) \text{ respectively.}$$

Hence
$$I + K + L = ng - \Theta.$$

Moreover the three must be equal, and accordingly:

$$I = K = L = \tfrac{1}{3}(ng - \Theta) \text{ must be an integer.} \quad (15)$$

When a polynomial is given, every term of which fulfils these conditions with the same g and Θ,—

$$(B_{IKL}^{\Theta}) = \Sigma B_{IKL}^{\Theta}; \quad (16)$$

then only the first step of Hilbert's process needs supplementing, in order to derive from (B_{IKL}^{Θ}) a complete covariant.

Solving the linear substitution (6), we find expressions for the x's in terms of the y's:

$$\left\{\begin{array}{l} (\xi\eta\zeta)x_1 = (y\eta\zeta) \\ (\xi\eta\zeta)x_2 = (y\zeta\xi) \\ (\xi\eta\zeta)x_3 = (y\xi\eta) \end{array}\right\}. \quad (17)$$

The first step (p. 4) will now be *to substitute in the polynomial* (B_{IKL}^{Θ}) *for the coefficients* a_{ikl} *their equivalents in terms of the* a_{ikl}, *for the several variables* $(x_1, x_2, x_3;$ $x_1', x_2', x_3'; \ldots .)$ *their equivalents in terms of transformed variables* $(y_1, y_2, y_3;$ $y_1', y_2', y_3'; \ldots .)$; *and to multiply the result by such a power of the modulus of substitution,* $(\xi\eta\zeta)$, *as will make the product an entire function of the nine parameters* ξ, η, ζ. The multiplier is obviously not of higher degree than the Θ^{th}; and as a superfluous factor here means merely a numerical factor in the final result, we may always employ

$$(\xi\eta\zeta)^{\Theta}.$$

After this first step the result may be symbolized by

$$(\xi\eta\zeta)^{\Theta}(\overline{B}_{IKL}^{\Theta}).$$

The second step is the same as in §1, save that Ω has an exponent slightly altered. Its effect is written symbolically:

$$\Omega^{\frac{1}{2}(ng+2\Theta)}\,.\,(\xi\eta\zeta)^{\Theta}\,.\,(\overline{B}_{IKL}^{\Theta}) = J(\bar{a}_{ikl};\ y). \tag{18}$$

Lastly there is to be included in the third step the replacement of variables (y) by variables (x); $J(a_{ikl};\ x)$ is then a covariant, and the process by which it is derived from (B_{IKL}^{Θ}) is perfectly determinate.

This process enables me to verify neatly an important theorem, which may be given here as a corollary. Suppose that $(B_{IKL}^{\Theta}) = F(a_{ikl};\ x,\ x')$ is a function of coefficients of a single groundform: a_x^n, and of only two sets of variables: $(x_1,\ x_2,\ x_3;\ x_1',\ x_2',\ x_3')$. Apply the operation described as the first step, and suppose further that an identity subsists as follows:

$$(\xi\eta\zeta)^{\Theta}(B_{IKL}^{\Theta}) \equiv \overline{F}(\bar{a}_{ikl};\ y,\ y') \equiv M\,.\,F(\bar{a}_{ikl};\ y,\ y') + G_1\,.\,\bar{a}_y^n + G_2\,.\,\bar{a}_{y'}^n \tag{19}$$

(see formulae 9); G_1 and G_2 denoting entire functions of both coefficients and variables, M a function of parameters ξ, η, ζ only. By the second operation we obtain evidently:

$$J(\bar{a}_{ikl};\ y,\ y') \equiv F(\bar{a}_{ikl};\ y,\ y')\,.\,(\Omega^{\frac{1}{2}(ng+2\Theta)}M) + \bar{a}_y^n\,.\,(\Omega^{\frac{1}{2}(ng+2\Theta)}G_1) + \bar{a}_{y'}^n\,.\,(\Omega^{\frac{1}{2}(ng+2\Theta)}G_2).$$

Hence the final result is a relation of the form:

$$J(a_{ikl};\ x,\ x') = M'\,.\,F(a_{ikl};\ x,\ x') + G_1'\,.\,a_x^n + G_2'\,.\,a_{x'}^n, \tag{20}$$

in which M' is a constant, G_1' and G_2' entire functions of coefficients and variables. This is stated as a proposition thus, adding obvious extensions of the above proof:

COROLLARY: *If an entire homogeneous function of the coefficients of several groundforms and of several cogredient sets of variables, of suitable weights, have the property that every linear transformation of the variables reproduces the function,* MODULIS *the given groundform; then it is possible to add to the function certain determinate multiples of the groundforms (written in the variables of each set separately), such that the sum is a covariant, i. e. reproduces itself with no additive terms upon every linear transformation of the variables.* Applications of this theorem arise in the study of point-systems upon algebraic curves.

CLARK UNIVERSITY, *April*, 1892.

Note on the Use of Supplementary Curves in Isogonal Transformation.

BY ROLLIN A. HARRIS.

---

1. The main object of this note is to show how the problem of representing one plane conformably upon another, using any real function* of the variable, may be made to depend upon the problem of constructing supplementary curves from given tracings of the corresponding principal curves.

2. It will be convenient to begin by determining the conditions for a monogenic function of the variable $x + kw$.

$\overline{X}$, W and x, w are real quantities; $k \equiv ij = \sqrt{1}$ where i, $j = \sqrt{-1}$. Suppose $\overline{X}$, W to depend upon x, w; then the ratio of the change in $\overline{X} + kW$ to the change in $x + kw$ will be

$$\frac{d\overline{X} + kdW}{dx + kdw}. \tag{1}$$

This ratio will be independent of the direction in which $d(x + kw)$ is taken, i. e. independent of the variable quantities dx, dw, if

$$\frac{\partial \overline{X}}{\partial x} = \frac{\partial W}{\partial w}, \quad \frac{\partial \overline{X}}{\partial w} = \frac{\partial W}{\partial x}. \tag{2}$$

It may be noted here that unless equations (2) are satisfied the quotient (1) will, in general, become infinite when the variable $x + kw$ is taken along one of the two systems of lines

$$w = \pm x + \text{constant}; \tag{3}$$

but when equations (2) are satisfied this difficulty disappears, for the quotient is then equal to $\frac{\partial \overline{X}}{\partial x} + k\frac{\partial W}{\partial x}$.

---

[*] The term "real function" will be applied to any monogenic function, $X + iY$, of $x + iy$ such that X remains unaltered when y is replaced by $-y$, and Y becomes $-Y$ by the same substitution.

3. Let ϕ denote any real function, and suppose $\phi(x + kw)$ to be developable by Taylor's theorem in powers of kw within certain regions of convergence; then

$$\phi(x) + \frac{w^2}{2\,!}\phi''(x) + \ldots = \cosh\left(w\frac{\partial}{\partial x}\right)\cdot\phi(x) = X, \tag{4}$$

$$w\phi'(x) + \frac{w^3}{3\,!}\phi'''(x) + \ldots = \sinh\left(w\frac{\partial}{\partial x}\right)\cdot\phi(x) = W, \tag{5}$$

which satisfy the conditions for a monogenic function.

4. *Isogonality in general.* Let X, Y and x, y denote, in this paragraph, real or imaginary quantities. Suppose X, Y to so depend upon x, y that

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}, \tag{6}$$

then at the intersections* of any two curves (real or imaginary),

$$y = f_1(x), \quad y = f_2(x), \tag{7}$$

we have

$$\tan^{-1}\left(\frac{dY}{dX}\right)_1 - \tan^{-1}\left(\frac{dY}{dX}\right)_2 \equiv \tan^{-1}\left(\frac{dy}{dx}\right)_1 - \tan^{-1}\left(\frac{dy}{dx}\right)_2, \tag{8}$$

where the subscript indicates the curve taken and multiples of π are disregarded. If an angle be measured in the contrary sense, as compared with the others, its sign in (8) must be altered. If a measurement start in advance of the initial direction by a certain amount, this amount must be added to one or the other of the members of (8).

Since X and Y each depend upon x, y, the relation $y = f(x)$ leads to a relation between X and Y. The curve, or portion of curve, (real or imaginary) representing this relation and corresponding to $y = f(x)$, is a complete image of $y = f(x)$.

If, now, we define

$$\tan^{-1}\left(\frac{dY}{dX}\right), \text{ or } \tan^{-1}\left(\frac{dy}{dx}\right),$$

as the angle which a tangent to a curve makes with the X-axis, or the x-axis, it follows that the difference between two such angles belonging to the intersecting images equals the difference between the two corresponding angles of the intersecting paths (7), regardless of the reality of the curves or angles. In this sense the angles are preserved.

---

* It is assumed that at these points, $\frac{\partial X}{\partial x}$ and $\frac{\partial Y}{\partial x}$ do not both vanish or both become infinite.

5. In like manner, if the variable point (x, w) be taken along any two intersecting curves, and if $\overline{X}$, W so depend upon x, w that

$$\frac{\partial \overline{X}}{\partial x} = \frac{\partial W}{\partial w}, \quad \frac{\partial \overline{X}}{\partial w} = \frac{\partial W}{\partial x},$$

we have, in general,

$$\tanh^{-1}\left(\frac{dW}{d\overline{X}}\right)_1 - \tanh^{-1}\left(\frac{dW}{d\overline{X}}\right)_2 \equiv \tanh^{-1}\left(\frac{dw}{dx}\right)_1 - \tanh^{-1}\left(\frac{dw}{dx}\right)_2, \tag{9}$$

where multiples of $j\pi$ are disregarded. If one of the hyperbolic angles,

$$\tanh^{-1}\left(\frac{dW}{d\overline{X}}\right)_1,$$

be reckoned from the W-axis, instead of from the $\overline{X}$-axis, and in the contrary sense, this relation becomes

$$\tanh^{-1}\left(\frac{d\overline{X}}{dW}\right)_1 - \tanh^{-1}\left(\frac{dW}{d\overline{X}}\right)_2 \equiv \tanh^{-1}\left(\frac{dw}{dx}\right)_1 - \tanh^{-1}\left(\frac{dw}{dx}\right)_2, \tag{10}$$

since

$$\tanh^{-1}\infty = \frac{j\pi}{2}, \quad \tanh^{-1}\frac{1}{a} = \frac{j\pi}{2} - \tanh^{-1}a,$$

and multiples of $j\pi$ are disregarded.

This form is useful when a real hyperbolic tangent is numerically greater than unity.

6. If y, $=f(x)$, be such a function of x that its values are real or purely imaginary for all real values of x, the entire curve $y=f(x)$ can always be drawn as a real curve in one or both of the two planes xy, xw where jw replaces y.

If both planes are required for the curve, the part in the one will be said to be supplementary[*] to the part in the other.

---

[*] Strictly speaking, the part in the one plane would have to be rotated into the plane of the other part before it could be called supplementary. For convenience this rotation may be left unperformed.

the plane xy,
If $y = f(x)$ lie in the plane xw,
the planes xy and xw,
the plane XY.
its complete image, by any real function ϕ, lies in the plane $\bar{X}W$.
the planes XY and $\bar{X}W$.

7. *Measurement of hyperbolic angles.* As a real angle whose tangent is known may be represented by twice the area of a circular sector, so a real hyperbolic angle whose hyperbolic tangent is known may be represented by twice the area of an hyperbolic sector. The radius of the circle is unity; the hyperbola is rectangular, having its semi-axis equal to unity. The area $(s/2)$ of the sector is connected with the true angle at the center of the hyperbola by the equation

$$\tan\theta = \tanh s.$$

This enables one to construct an hyperbolic protractor whose divisions represent equal areas.

8. If the variable $x + kw$ describe any two real intersecting paths in the xw-plane, and $\bar{X} + kW$ describe two real intersecting curves in the $\bar{X}W$-plane, the angles of intersection will be preserved—all angles being measured with an hyperbolic protractor, and in accordance with §§4, 5.

All straight lines cutting the x-axis at a true angle of ± 45° transform into straight lines cutting the $\bar{X}$-axis at a true angle of ± 45°. If the xw-plane be divided into rectangles by the pair of systems $w = \pm x +$ constant, the $\bar{X}W$-plane (or at least a certain portion of it) will also be divided into rectangles by the corresponding systems $W = \pm \bar{X} +$ constant.

9. *Quasi images.* The result of transforming any curve or point in the xw-plane to the $\bar{X}W$-plane, by means of a monogenic function, may be called the quasi image of that curve or point.

The last remark in §8 enables one to construct mechanically the quasi image of any given path in the xw-plane, as soon as he has computed the real values of the transforming function corresponding to the real values of the variable x. For, the pairs of systems there mentioned determine all corresponding points in the two planes.

If $\bar{X}$ and W are each symmetric in x, w, then the quasi image of $x=f(w)$ has the same equation as has that of $w=f(x)$. If $\bar{X}$ is the same function of x, w as W is of w, x, then the equation of the quasi image of $x=f(w)$ may be obtained from that of $w=f(x)$ by interchanging $\bar{X}$ and W.

10. Application to the construction of curves supplementary to the true images of certain paths symmetric with respect to the x-axis.

If $y=f(x)$ denote any real curve symmetric about the x-axis, its true image, whose equation is got by eliminating x and y from

$$y=f(x)$$
$$X+iY=\phi(x+iy),$$

is symmetric about the X-axis, ϕ being a real function.

Now suppose f to be such a function that for certain real values of x, y becomes a pure imaginary. Replacing y by jw (§6), we obtain a real locus, $jw=f(x)$, supplementary to the real part of $y=f(x)$. The quasi image of this is got by eliminating x and jw from the equations

$$jw=f(x)$$
$$\bar{X}+ijW=\phi(x+ijw)$$

since $k=ij$. But this resultant is the same function of $\bar{X}$, jW as the former resultant was of X, Y; hence the following:

*The quasi image of the supplement of a given path is supplementary to the true image of the same path.**

(In representing a region of the xy-plane conformably upon the XY-plane, it is usually best to select very simple systems of curves for paths of the variable. Having thus determined corresponding points in the two planes, the image of any given path can be drawn at once.)

11. *Special case.* The quasi image of $x=\alpha$ is supplementary to the true image of $x=\alpha$.

If the function $\phi(x)$ have a focus† upon the X-axis, this point is a real focus of the true image of the system $x=$ constant‡. For, the quasi image of the line

---

* The $\bar{X}$-axis is now supposed to coincide with the X-axis.

† i. e. a point in whose immediate vicinity the transformed elements are indefinitely small in comparison with their former magnitude.

‡ The nature of ϕ, and the particular focus chosen, when there are several such foci, determine the limits between which this constant may be taken.

$x = \alpha$ (because it crosses the X-axis and passes through opposite angles of the elementary rectangles into which the XW-plane is divided by the systems $W = \pm X +$ constant) must touch the two lines in the XW-plane passing through the focus of the function making angles of ± 45° with the X-axis. Now these two lines, when referred to the XY-plane, still pass through the focus of the function; they also pass through the circular points at infinity, and are imaginary lines tangent to the true image of $x = \alpha$, which is supplementary to the quasi image of $x = \alpha$.

Example. If $\phi \equiv$ sine, the quasi image of the system $x =$ constant is a system of ellipses inscribed in a square having two of its opposite corners at the points $X = \pm 1$, $W = 0$ which are the foci of $\phi(x)$. Consequently the true image of the system $x =$ constant is a system of confocal hyperbolas having the points $X = \pm 1$, $Y = 0$ for foci. To construct the system of ellipses mechanically (§9), lay off upon the X-axis the values of $\sin x$ as x varies uniformly from $-\pi/2$ to $+\pi/2$. Draw two systems of straight lines cutting the X-axis at angles of ± 45°, and join the opposite corners of the elementary rectangles thus formed; the curves are the required quasi image of the system $x =$ constant by the function ϕ.

12. Suppose

$$X + kW = (x + kw)^2,$$

and let $x + kw$ move over the straight lines

$$x = \alpha, \quad w = mx;$$

$X + kW$ will describe a parabola, and a straight line passing through the focus of the supplementary parabola.

∴ Given any parabola of the second order whose axis of symmetry is made to coincide with the X-axis, and any secant line passing through the focus of the supplementary curve; let tangents be drawn at the points of intersection: the one will be inclined as much to the X-axis as the other will be to the W-axis,—the angles being measured in opposite directions (§§4, 5, 7).

If

$$X + kW = (x + kw)^n,$$

where n is any integer greater than unity, and $x + kw$ move as before, the above statement holds true if the words "any parabola of the second order" be replaced by "certain curves of the nth order."

If

$$X + kW = (x + kw)^{\frac{1}{2}}$$

and $x + kw$ move over the straight lines

$$x = \alpha, \quad w = \delta,$$

$X + kW$ will describe a circle and a rectangular hyperbola. If they intersect, let tangents be drawn to the two curves at a point of intersection. The tangent to the circle cuts the one asymptote at the same angle as the tangent to the hyperbola does the other.

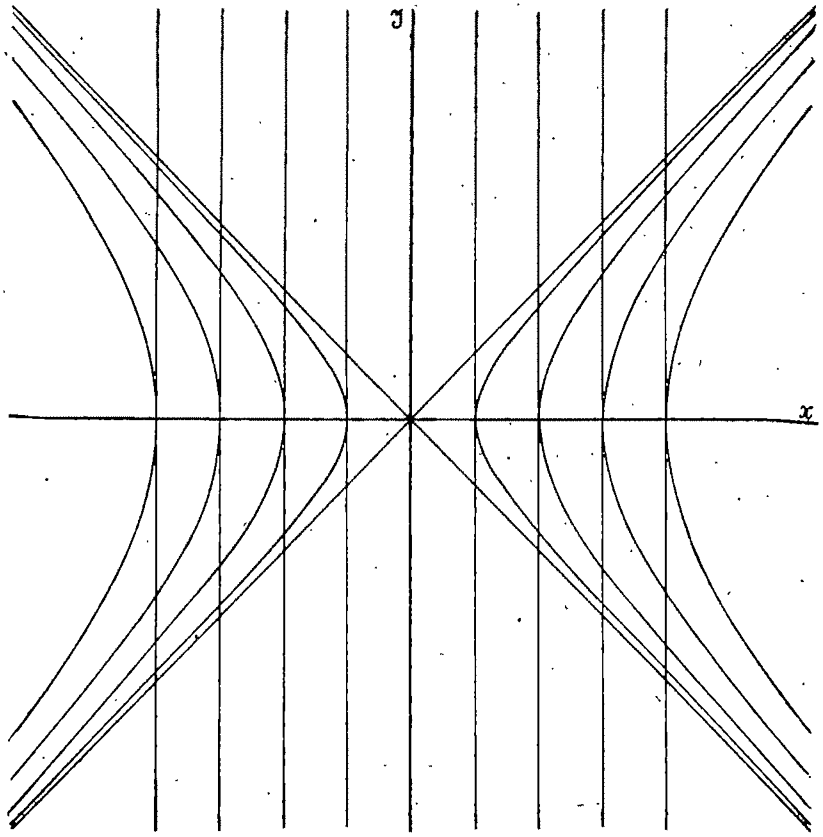


FIG. 1.

13. The accompanying figures illustrate the simple case where

$$X + iY = (x + iy)^2,$$

and where the variable is taken over the systems $x =$ constant and

$x^2 - y^2 = (\text{constant})^2$. Of course $X + iY$ will describe a system of confocal parabolas and a system of straight lines parallel to the Y-axis.

Figs. 1 and 2 may remain the same for various transforming functions; Fig. 3 is obtained from Fig. 2 by mechanical construction (§9), $\phi(x)$ having been laid off upon the X-axis.

The problem which this note leads up to, is that of passing in general from Fig. 3 to Fig. 4; i. e. from certain given curves to their supplementaries.

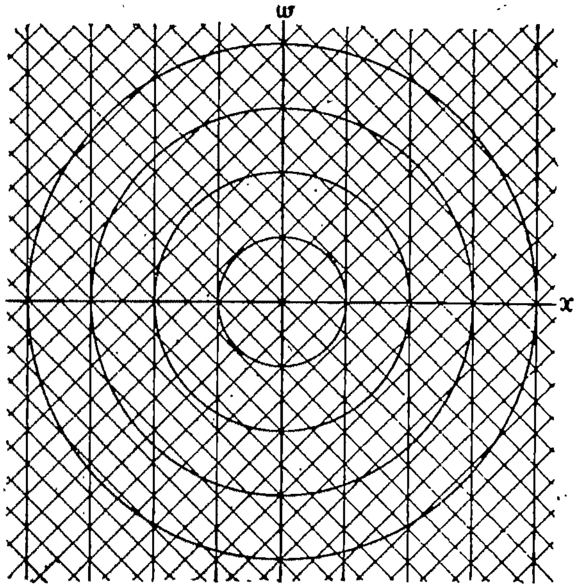


FIG. 2.

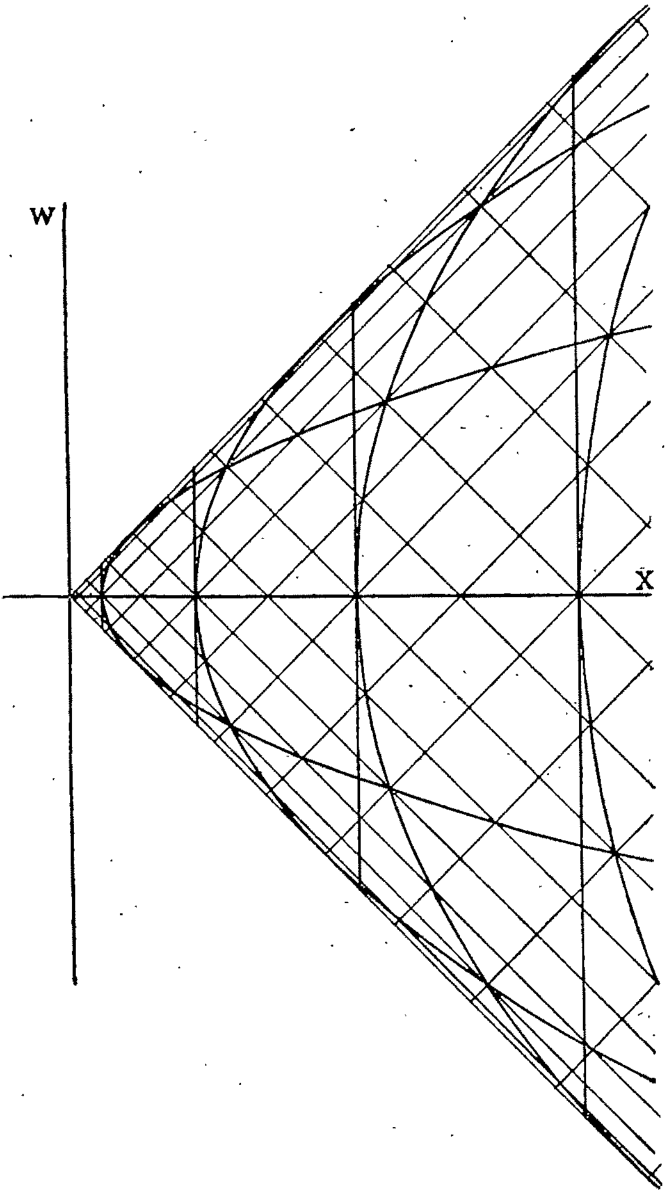


FIG. 3.
Scale ½ that of Figs. 1 and 2.)

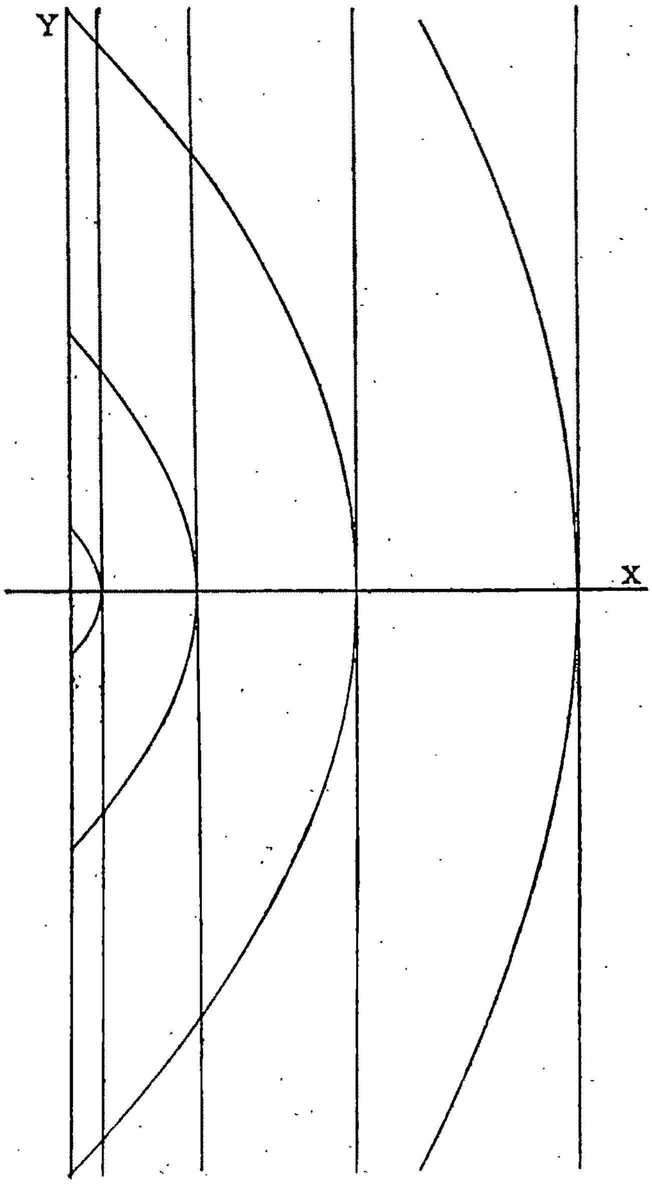


FIG. 4.
(Scale ¼ that of Figs. 1 and 2.)

On the Higher Singularities of Plane Curves.

BY CHARLOTTE ANGAS SCOTT.

1. The question of the analysis of Higher Singularities is one whose importance has been recognized from the time of Cramer. Two of the principal methods of dealing with it—by expansions, and by successive quadric transformations—are used in his "Analyse des Lignes Courbes"; the explanation of the principles involved, and the development of the theory, belong to the present day. The memoirs to which reference is made in this paper are those by Cayley, "On the Higher Singularities of a Plane Curve," 1866 (Quarterly Journal, VII, 212), and H. J. Smith, "On the Higher Singularities of Plane Curves," 1873–6 (Proc. of the London Math. Soc. VI, 153), and the series of papers by Brill and Nöther in the Mathematische Annalen, IX (1876), XVI (1880), XXIII (1884), etc.

2. The question consists of two principal parts. In the first place, in the case of any composite singularity the methods enable us to assign numbers δ, $\varkappa$, for the nodes and cusps that would produce the same deficiency and the same reduction in the class of the curve as the singularity in question. But these being determined, there is then the geometrical side to be considered. Is there any geometrical reality corresponding to this algebraic symbol? Do the numbers δ, $\varkappa$, express actual facts, or are they a conventional representation of the point to satisfy Plücker's equations? i. e. having determined that a certain compound singularity is equivalent to δ nodes, $\varkappa$ cusps, etc., is there a penultimate form in which these singularities exist indefinitely near together? This question is to a certain extent resolved by Cramer, in special examples, by means of the transformation $y = vx$;[*] he shows, for instance, that certain singularities with coin-

---

[*] It is noticeable that Newton uses this transformation, $y = xy$, in his Enumeratio Lin. Ter. Ord., calling the curve so obtained a hyperbolism of the original curve.

cident tangents occur as the final form of singularities with distinct tangents, involving a number of evanescent loops; the ordinary cusp thus presents itself as a node with an evanescent loop. Taking, e. g. (p. 636) the curves

$$x^5 = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4, \tag{1}$$

Cramer applies the transformation ($y = ux$)

$$x = x_1, \ y = x_1y_1.$$

This gives us a new curve,

$$x_1 = a + by_1 + cy_1^2 + dy_1^3 + ey_1^4; \tag{2}$$

both curves being referred to the same axes we have a simple construction for determining corresponding points. The intersections of (2) with the axis of y determine the passages of (1) through the origin; an arc of (2) cut off by the axis of y corresponds to a loop of (1) closed at the origin. If, e. g., all four roots of the equation $a + bz + cz^2 + dz^3 + ez^4 = 0$ coincide, (2) has a 'serpentement' presenting the appearance of ordinary contact; (1) has at the origin a point that presents the appearance of a simple cusp; but the cusp contains three vanishing loops, and is really a quadruple point. Again, on p. 638, Cramer uses a higher transformation

$$x = x_2y_2, \ y = x_2y_2^2,$$

by means of which he reduces

$$x^6 + ax^4y + cx^3y^2 + dxy^3 + ey^4 = 0 \tag{3}$$

to depend on the conic

$$x_2^2 + ax_2 + c + dy_2 + ey_2^2 = 0. \tag{4}$$

This transformation is of course equivalent to the two

$$x = x_1, \ y = x_1y_1; \ x_1 = x_2y_2, \ y_1 = y_2;$$

but Cramer uses it in the form first given. Intersections of (4) with either axis indicate passages of (3) through the origin, the interpretation differing somewhat for the two axes; and by considering the different possible varieties and positions of the conic (4), Cramer obtains all the varieties of the quadruple point on (3).

3. These same two transformations (i. $y = vx$, ii. $x = vy$) are used by Nöther in his paper "Ueber die singulären Werthsysteme einer algebraischen Function," etc. (Math. Ann. IX, 166), which is devoted to the development of

the analytical theory; his conclusion is that any compound point-singularity, a multiple point of order k, is made up of a simple k-point, with, in directions indicated by coincident tangents, other multiple points of orders $\not> k$; with a number V of branch points, which with the same number of double points form V cusps; so that for the resolution of the singularity we have

$$\delta = \sum \frac{k_1(k_1-1)}{2} - V; \quad \varkappa = V.$$

4. This theory is presented by Nöther as a purely analytical one, involving no dependence on geometrical ideas even when geometrical terms are used. But by this exclusion of geometry the investigation does not gain in rigor, as regards the analysis of higher singularities of algebraic curves; and it fails to show very clearly the existence of the various elements of the compound singularity. The actual significance of the transformation and resolution appears at once when we make use of the geometrical method of Quadric Inversion, described by T. A. Hirst in the Proceedings of the Royal Society, March, 1865. For convenience and completeness I shall reproduce, with obvious extensions, such of Dr. Hirst's results as are necessary for the main object of this paper, the Graphical Analysis of Higher Singularities. There is of course no occasion for this purpose to recognize any non-projective distinctions.

5. We have a fixed origin O, and a fixed base conic. Points collinear with regard to O, conjugate with regard to the conic, are said to be *inverse*. If OI, OJ are the tangents from O to the conic, the inverse of a straight line is a conic through OIJ; the correspondence is therefore a 1 . 1 quadric correspondence, with O, I, J, as fundamental points, and we have different cases, depending on the relative position of O, I, J; (1) if the conic be a proper conic, and O not on it, the three fundamental points are distinct; (2) the conic may be a line pair, and O not on it, which implies the coincidence of I, J; (3) the conic being a proper conic, O may be on it, in which case I, O, J, are consecutive points on a curve of finite curvature; or finally (4) the conic may degenerate, and O may be on it, this happens when OIJ are collinear; this case need not be considered, as it is simply 'harmonic projection.' It is convenient to call the polar of O with regard to the conic the *base line*.

If the curve to be inverted passes through any of the fundamental points O, I, J, then the fundamental lines IJ, OI, OJ, will present themselves in the inverse; they are, however, to be rejected, and the residuum is to be counted as the *proper* inverse. Thus, e. g., the inverse of a line OP through O appears as $OP\,.\,IJ$. Rejecting IJ, we have: the inverse of a line through O is the line itself; and similarly, if IJ are distinct, and A a point on the base conic, the inverse of IA is JA. This gives a convenient construction for the inverse P' of a point P; let IP cut the base conic in A, then JA meets OP in P'.

6. From the definition it appears at once that the inverse of any point on the base line is the origin, and that consequently k intersections of a curve with the base line give, on the inverse, a k-point at the origin. Similarly, in case (1) for the relation of I to OI, and of J to OJ. Thus a curve of order n, with points of multiplicity i, j, k at I, J, O, gives a curve of order n', with points of multiplicity i', j', k'; where

$$\begin{aligned} n' &= 2n - i - j - k, \\ i' &= n - i - k, \\ j' &= n - j - k, \\ k' &= n - i - j. \end{aligned}$$

If however we take IJ coincident (case (2)), then an intersection with OI gives a branch touching the base line; thus corresponding to points on OI we get a number of branches with contact at I. And if we take IOJ as in case (3), we get branches having three consecutive points common.

Thus the inverse of a conic *not* passing through the fundamental points is

(i) a quartic with dps at O, I, J;
(ii) a quartic with a dp at O, and a tacnode at I;
(iii) a quartic with an oscnode at O.

7. The formulæ of transformation are obtained at once; for the three cases we get

$$(1)\ x : y : z = x'z' : y'z' : x'y';$$
$$(2)\ x : y : z = x'z' : y'z' : x'^2;$$
$$(3)\ x : y : z = x'^2 : x'y' : -(x'z' + 2my'^2);$$

to a projection of one or other of which, as is well known, the most general 1 . 1 quadric transformation can be reduced.

8. Applying the method to the analysis of higher singularities, we notice in the first place that a multiple point of any order, with a number of coincident

or distinct tangents, gives in the inverse, if not on a fundamental line, a multiple point of the same nature. Again, since a straight line not passing through a fundamental point becomes a conic, we see that inflexions may be gained or lost; and similarly for double tangents. The method, then, is *directly* applicable to the consideration of singularities *only as point-singularities.*

9. Attending only to cases (1), (2), the base line does not pass through the origin. Let a branch cut the base line in Z; to study the inverse it is convenient to mark off and number corresponding divisions (Fig. (1)).

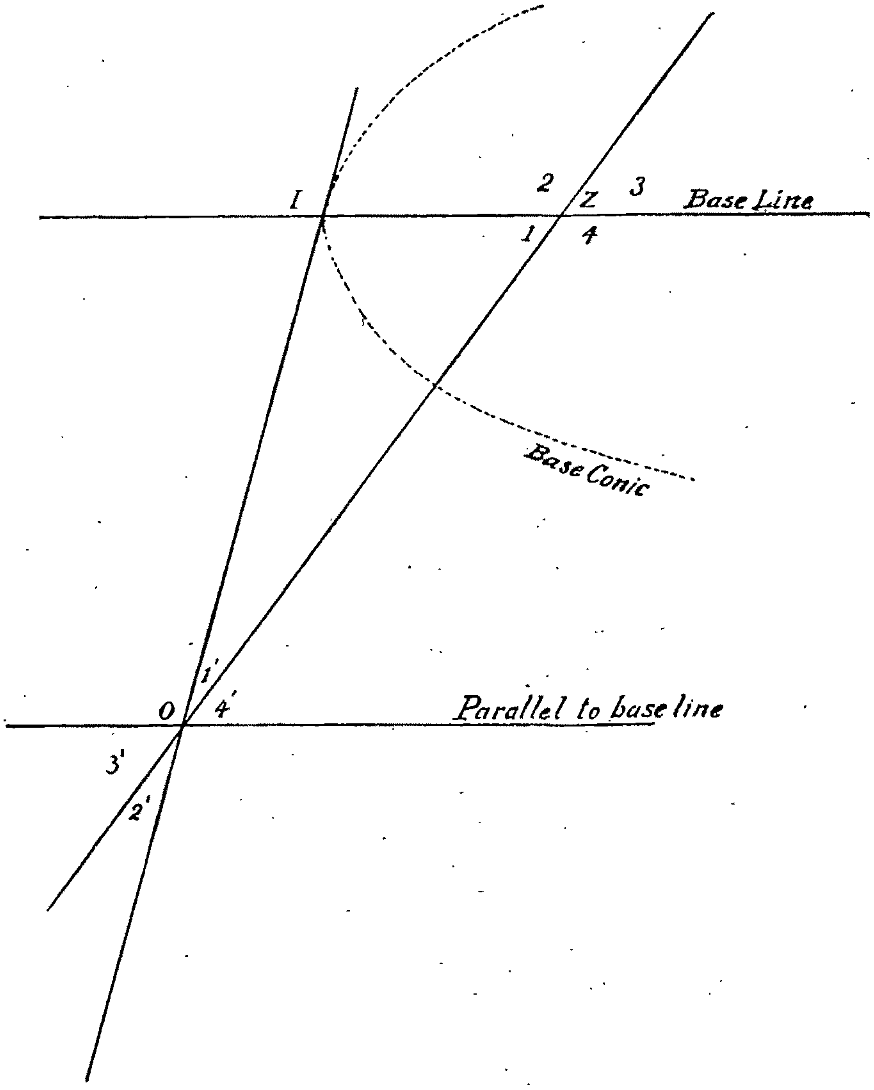


FIG. 1.

Let the branch cutting the base line at Z pass from 1 to 3; its inverse has therefore to pass from 1′ to 3′ through O, and so describes a branch through O, having OZ as the tangent there. If, however, the branch through Z has OZ as the tangent, it passes from 1 to 2, and its inverse, passing from 1′ to 2′, and touching OZ at O, has an inflexion there; and in general a branch having with OZ at Z contact of order n gives a branch having with OZ at O contact of order $n+1$.

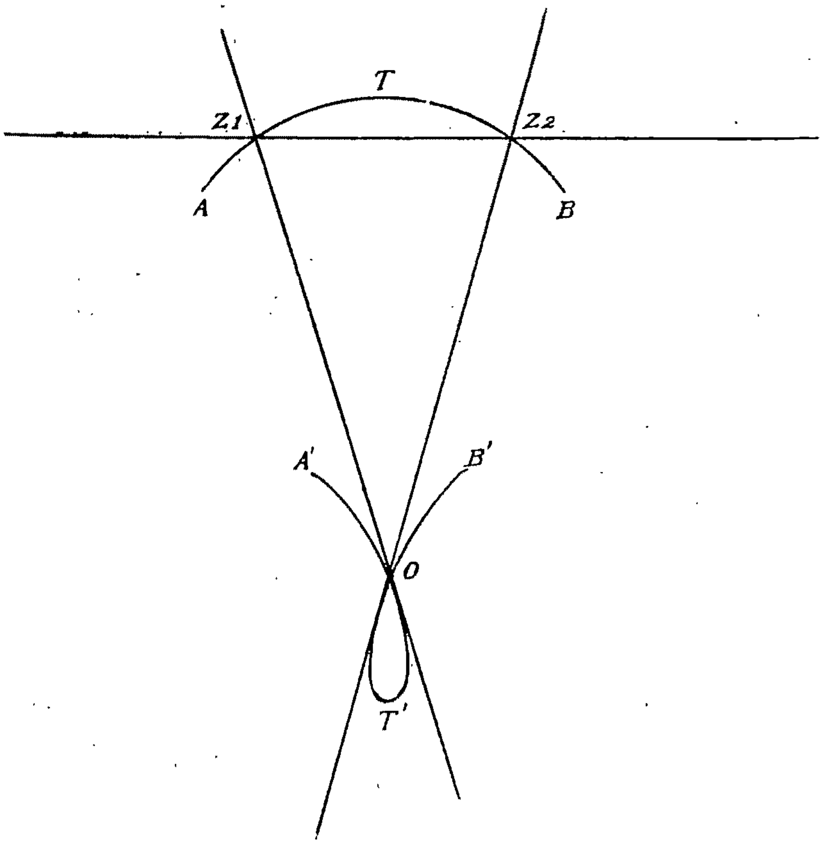


FIG. 2.

10. Next let the curve cut the base line in two points Z_1Z_2 (1) on the same branch. The inverse of the arc Z_1TZ_2 (Fig. (2)) is the loop $OT'O$, T' being inverse to T. If then the points Z_1Z_2 close up, the arc included vanishing, so that we have ordinary contact with the base line, the loop $OT'O$ closes up, and is finally evanescent, with coincident tangents; we have then, corresponding to a branch in 1, 4 with contact at Z, a branch in 1′, 4′, having contact with OZ at O; we

have; that is, a cusp (Fig. (3)). The simplest cusp, then, presents itself as an evanescent loop with a node.

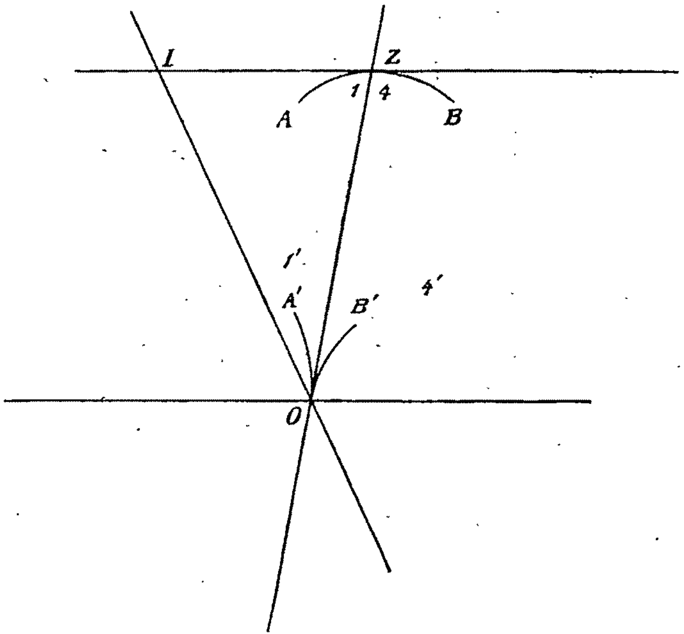


FIG. 3.

(2) But the points Z_1Z_2 need not be on the same branch; they may coincide owing to the presence of a double point at Z. In this case we have a tacnode at O. To see how this arises, consider the penultimate form, with the double point N not on the base line, but indefinitely near to it (Fig. (4)). We have

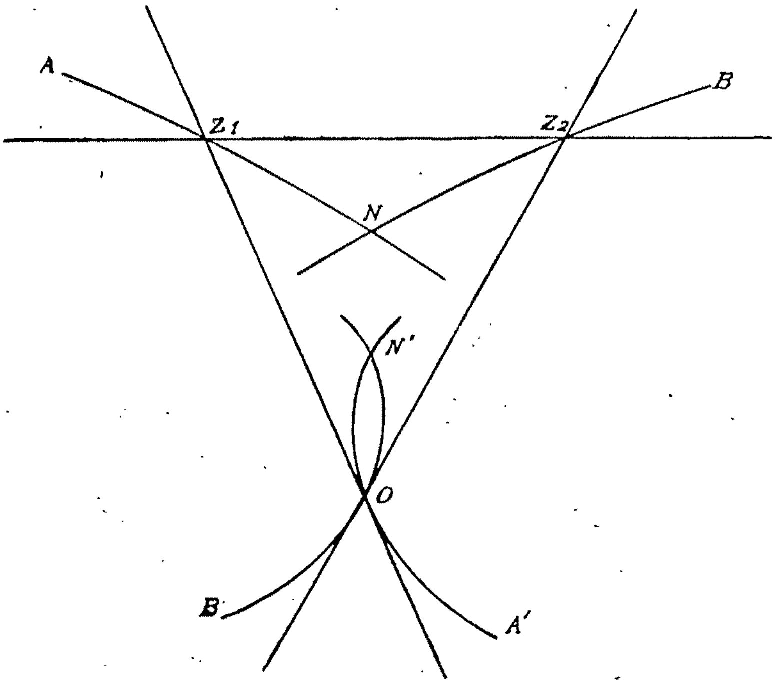


FIG. 4.

then in the inverse a node O, due to Z_1Z_2, and a node N', due to N. Thus we get a tacnode, Fig. (5), with OZ as tangent. The double point at O, then, on inversion, sheds one double point, and preserves the other.

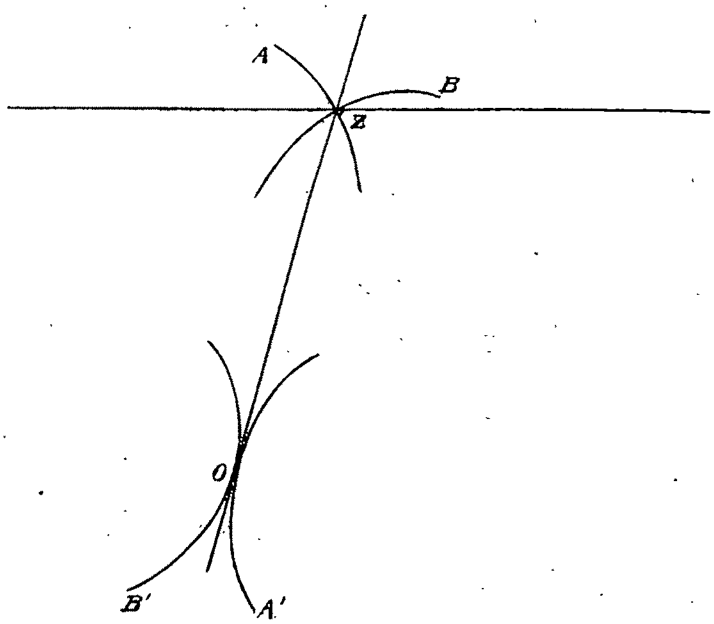


FIG. 5.

(3) Again, the points Z_1Z_2, being on the same branch, may coincide on account of a cusp. Now we already know that a cusp arises from an evanescent loop with a node; if then the cuspidal tangent does not pass through O, we have the resolution as in Figs. (6), (7), the cusp giving rise to a ramphoid cusp, whose

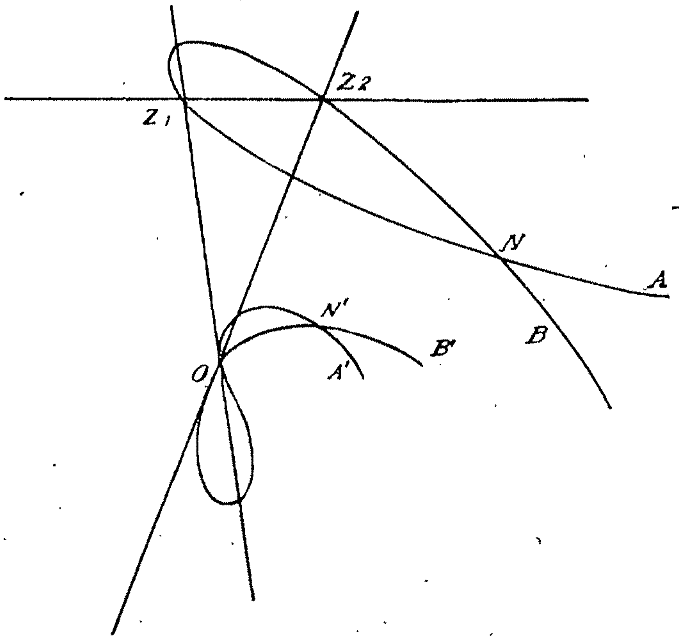


FIG. 6.

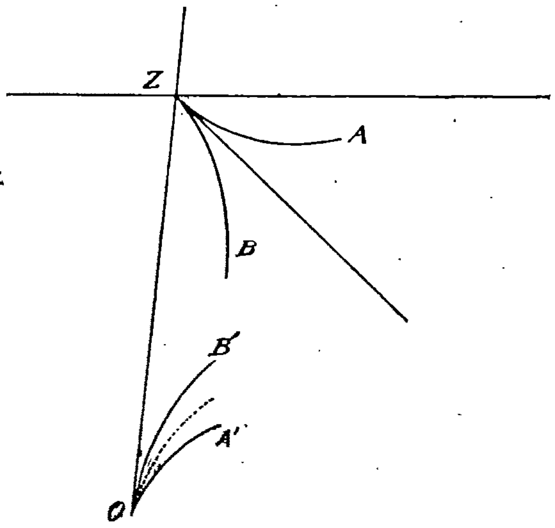


FIG. 7.

resolution into two nodes and an evanescent loop is here shown. It may be noticed, too, that the presence of the inflexion in the ramphoid cusp is shown by the one tangent from O to the evanescent loop NZ_1Z_2. If, however, the cuspidal tangent passes through O, it inverts, not into the conic of curvature as in Fig. (7), but into a straight line, and the inverse of the ceratoid cusp is a ceratoid cusp in appearance, but it is equivalent to a simple cusp with an extra node, i. e. to $y^2 = x^5$ (Figs. (8), (9)), and involves two inflexions, indicated by there being two tangents from O to the loop NZ_1Z_2.

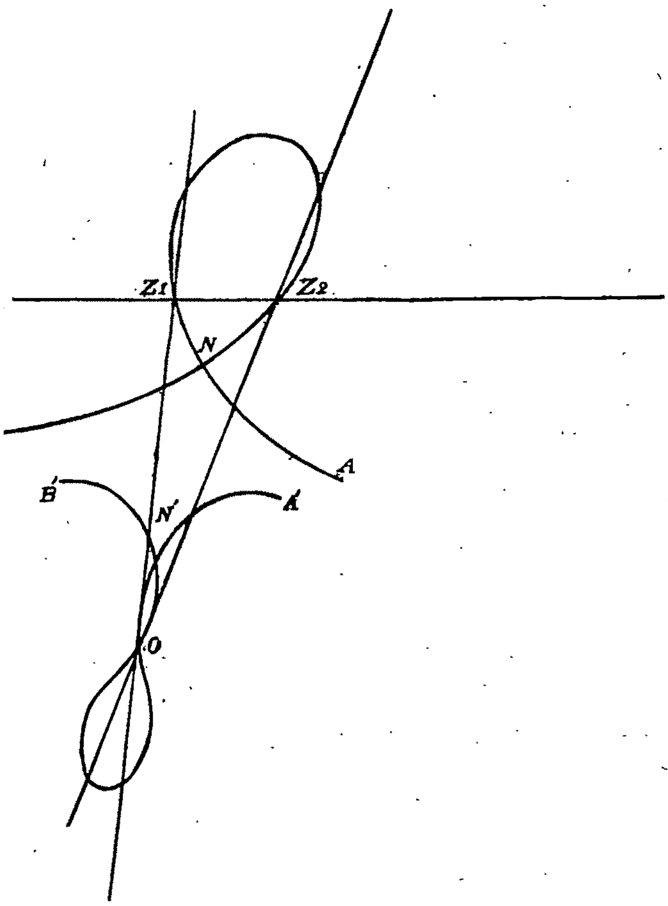


FIG. 8.

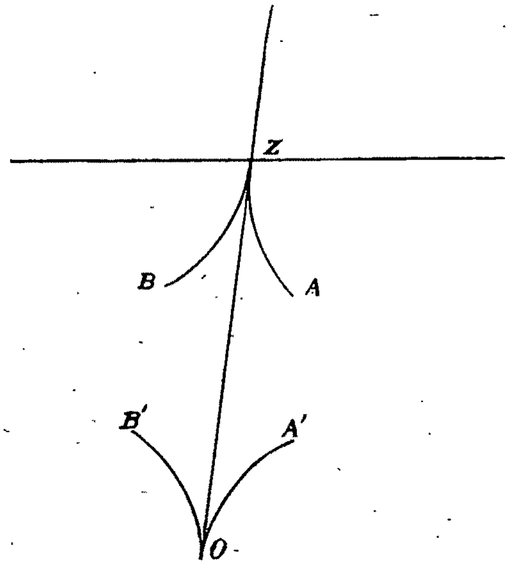


FIG. 9.

11. If now we have *three* coincident points on the base line, we get at O a triple point, differing according to the cause of the threefold coincidence. This may be (1) an inflexion. Resolving this in the recognized way, we see that the

invisible singularity resulting is really (Figs. (10), (11)) a triple point with two evanescent loops, i. e. it is equivalent to one node and two cusps (Cramer, p. 610).

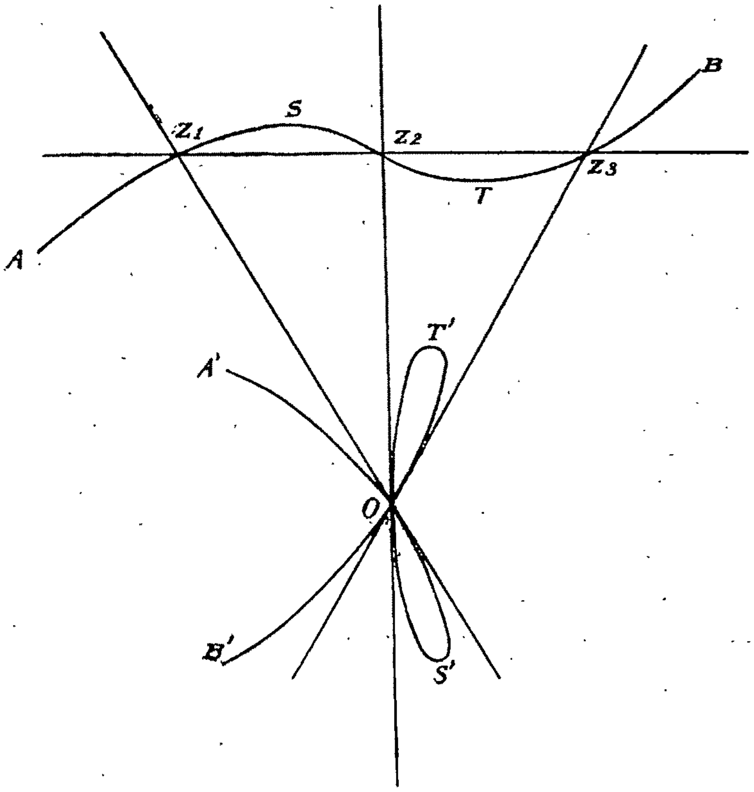


FIG. 10.

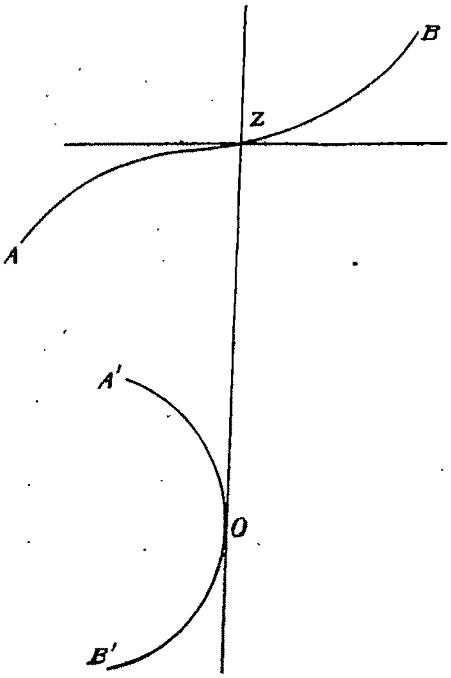


FIG. 11.

(2) There may be on the base line a node, the base line being a tangent; displacing this slightly, we see that the inverse has a cusp with an ordinary branch passing through it; the singularity is a triple point O, with a neighboring double point N', and it contains one cusp (Figs. (12), (13)).

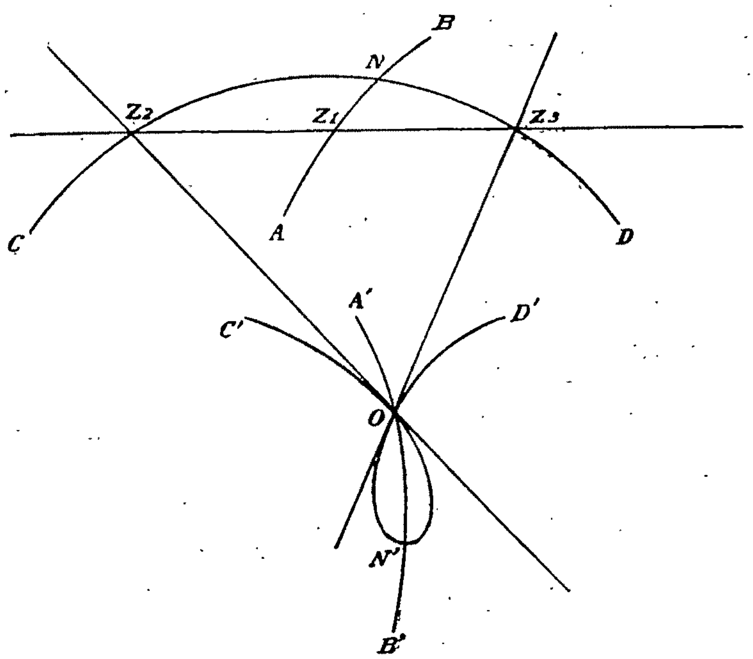


FIG. 12.

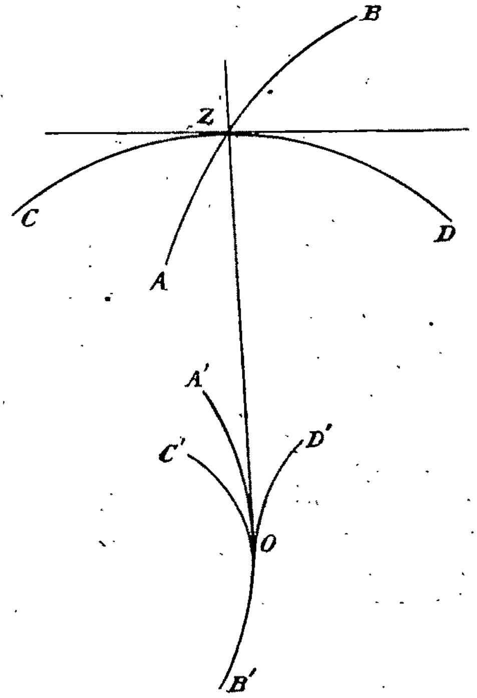


FIG. 13.

(3) There may be a cusp with the base line as tangent. Resolving this (Figs. (14), (15)) we have two evanescent loops, combined with a triple point O and a double point N'. The singularity, then, while presenting the appearance of an inflexion, gives $\delta + \kappa = 4$, $\kappa = 2$.

Displacing the components of the triple point O, we have Fig. (16).

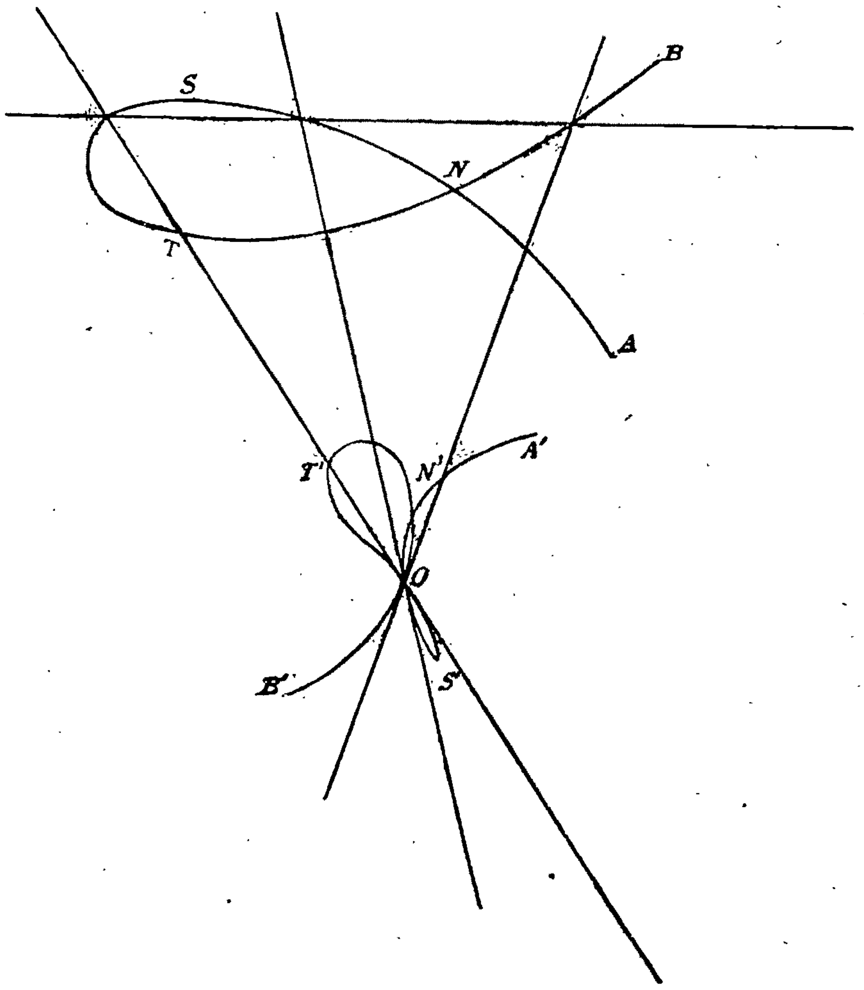


FIG. 14.

(4) If the threefold coincidence on the base line is due to a triple point, this presents itself as a triple point in the immediate neighborhood of O; we have then two consecutive triple points.

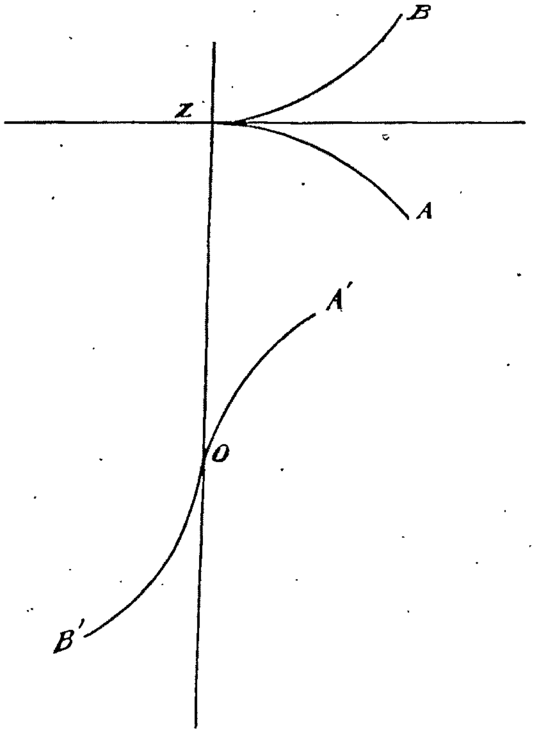


FIG. 15.

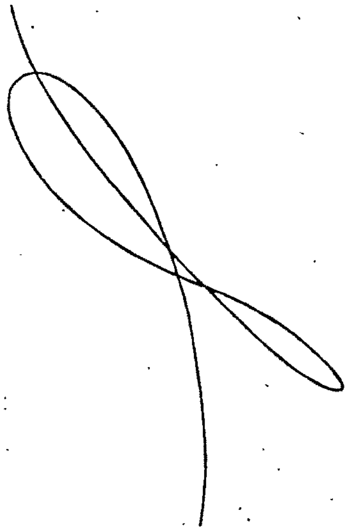


FIG. 16.

12. Reversing the process, we see that a singularity at O, when inverted, sheds the multiple point of highest order k, this being represented by k points on the base line; some or all of these will coincide if the singularity has coincident tangents, and for each such coincidence we have on the base line a multiple point of order k' ($k' \ngtr k$). The original singularity of order k is therefore equivalent to a simple k-point, together with singularities of orders k', k'', etc. ($k' \leqq k$). Similarly in dealing with the intersections of two curves, at a point which is an i-point on one, a k-point on the other: the obvious number of intersections is ik; but if any of the tangents are the same for the two curves, there will be neighboring intersections. To detect these, we clear away the ik already found, by means of a single inversion (Nöther, Math. Ann., XXIII, 323), and the next group of intersections is thus exposed.

13. Returning to the singularity on the one curve, let the order $\geqq k$, and let there be k coincident tangents; we have then on the base line k coincident points.

We have, that is, a singularity of order k', $k' \not> k$. If $k' < k$, the base line is a tangent to one or more branches, and therefore a certain number of evanescent loops present themselves, indicating in the resolved singularity the presence of a certain number of cusps.

These cusps necessarily present themselves when a k-point has to be made by l branches, $l < k$. Thus, e. g., if one branch has to make a triple point, it must turn back twice; and there are two cusps. In general, for a single branch to make an α-point, the inverse—also a single branch—must meet the base line in α points, giving contact of order $\alpha - 1$, and we have therefore $\varkappa = \alpha - 1$ (H. J. Smith, L. M. Soc., VI, 161, 170).

14. The geometry for the analysis of a singularity is the same whether IJ are distinct or coincident. For special lines of reasoning it may be convenient to take IJ separate, e. g. in H. J. Smith's paper the curves are generated by two pencils whose vertices may be conveniently taken for I, J. But in transformation it is essential that the branch to be inverted shall *not* touch OI (or OJ); and yet the transformations are more conveniently followed if the tangent is a side of the coördinate triangle. Thus the second form of the transformation, that with IJ coincident, is the most convenient for the analysis of singularities.

15. The formulæ are

$$x : y : z = x'z' : y'z' : x'^2,$$

the singularity being at xy with $y = 0$ as the tangent, and the inverse at $y'z'$. If then we use Cartesian coördinates x, y, 1, and write $x' : y' : z' = 1 : y_1 : x_1$, the singularity is at xy and its inverse at x_1y_1; the formulæ are $x = x_1$, $y = x_1y_1$, which is the Cramer-Nöther transformation. Here $y = 0$ (the same line as $y_1 = 0$) is a tangent; $x = 0$ is any line through the singular point, inclined at a finite angle to the tangent considered—call this the *axis*; $x_1 = 0$ is the third side of the fundamental triangle, the *base*. Thus the singularity is inverted along a tangent, from its intersection with the axis to its intersection with the base.

The process simply splits up the compound singularity into a number of simple singularities; i. e. it shows how the singularity with coincident tangents is made up of singularities with distinct tangents. No application of the method of quadric inversion will replace a k-point by the equivalent $\frac{1}{2}k(k-1)$ nodes; the most that it can do is to separate the multiple points (Nöther), and replace the cusps by evanescent loops. The simple k-point may then, however, be further resolved by displacement (e. g. Brill, Math. Ann., XVI).

16. As an example, take the curve

$$x^4 y^3 = z^7,$$

which has compound singularities at xz, yz.

$$x^4 = z^7$$

gives as first inverse $$x_1^4 = z_1^3.$$

Here the base is the tangent at a triple point. Inverting along this tangent, we get for the second inverse

$$x_2 = z_2^3;$$

so the point $x_1^4 = z_1^3$ is that illustrated in Figs. (10), (11), a triple point with two evanescent loops. Inverting this, we get the original point, a quadruple point with a neighboring triple point,[*] these implying 9 nodes and cusps. The presence of the three loops shows that there are three cusps; we have therefore

$$\delta = 6 \, , \; \varkappa = 3$$

(Figs. (17), (18)).

The other singularity, $y^3 = z^7$, gives as the first inverse $y_1^3 = z_1^4$, the same point, but with the base line *not* a tangent. Figs. (19), (20) show that the original singularity is made up of two triple points, involving two cusps: i. e. $\delta = 4$, $\varkappa = 2$.

The curve $x^4 y^3 = z^7$, which presents the appearance of Fig. (21), is therefore, in its resolved form, represented by Fig. (22).[†]

---

[*] In Fig. 17, the elements of the triple point are drawn separated by the base line; in the original the triple point therefore presents itself as three nodes.

[†] That the numbers obtained by this process agree with those obtained by means of the expansions appears at follows. Let the branch be made up of k partial branches, giving expansions

$$y = x^{\alpha_1} + x^{\beta_1} + \dots .$$
$$y = x^{\alpha_2} + x^{\beta_2} + \dots ;$$

the transformation $y = xy$ reduces these to

$$y = x^{\alpha_1 - 1} + x^{\beta_1 - 1} + \text{ etc.}$$

Every exponent, and consequently the exponent at which any two expansions separate, is reduced by unity; i. e. since there are k expansions, twice the number of intersections is reduced by $k(k-1)$, and therefore the singularity sheds a point of order k, etc.

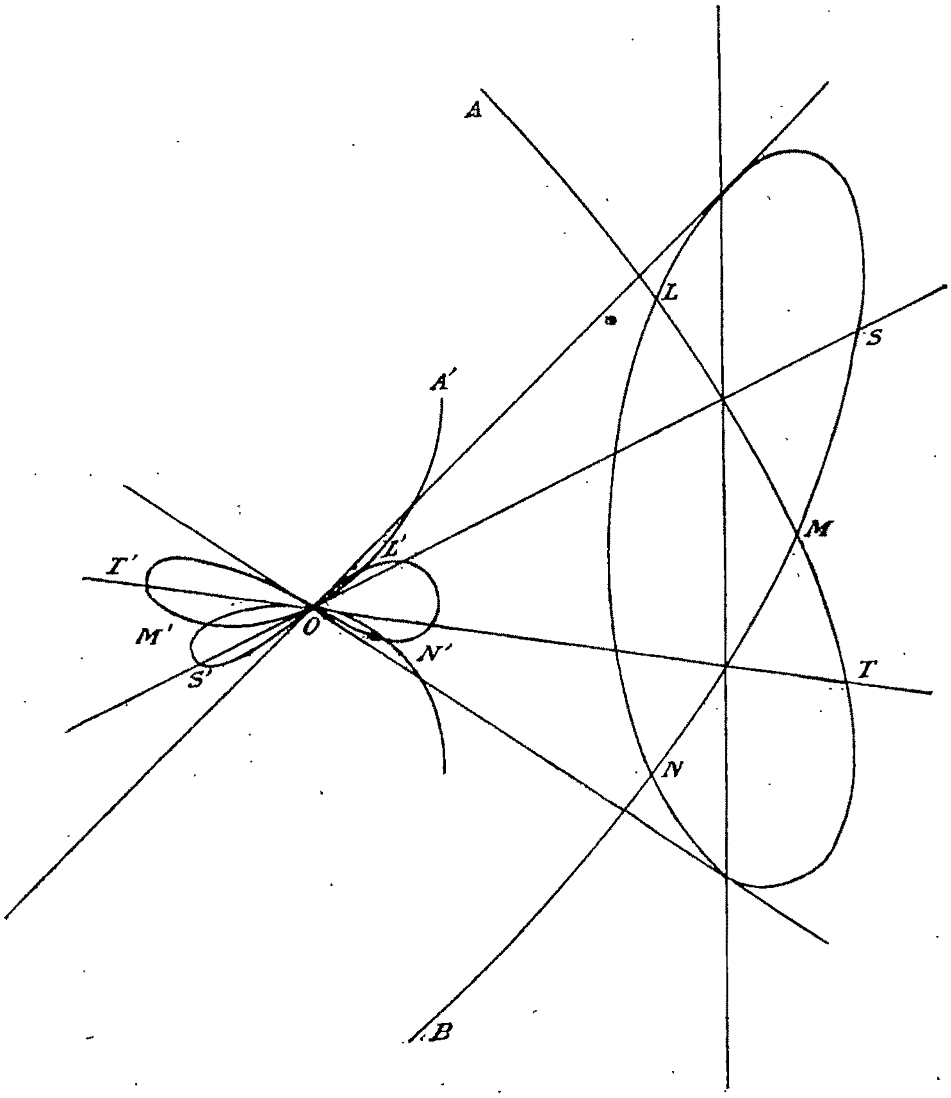


FIG. 17.

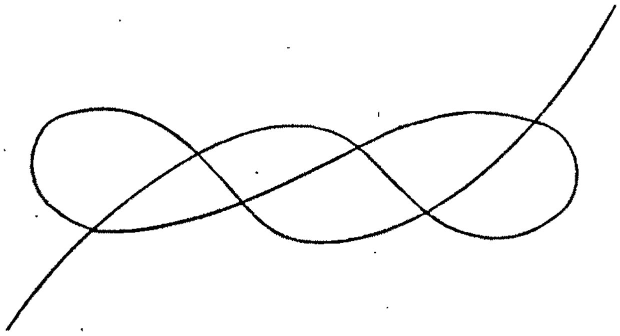


FIG. 20.

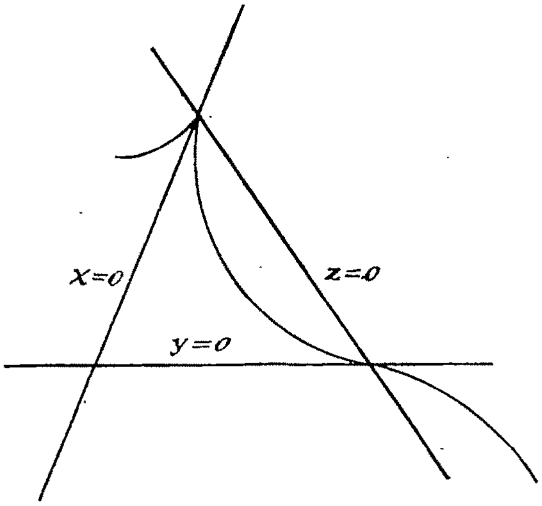


FIG. 21.

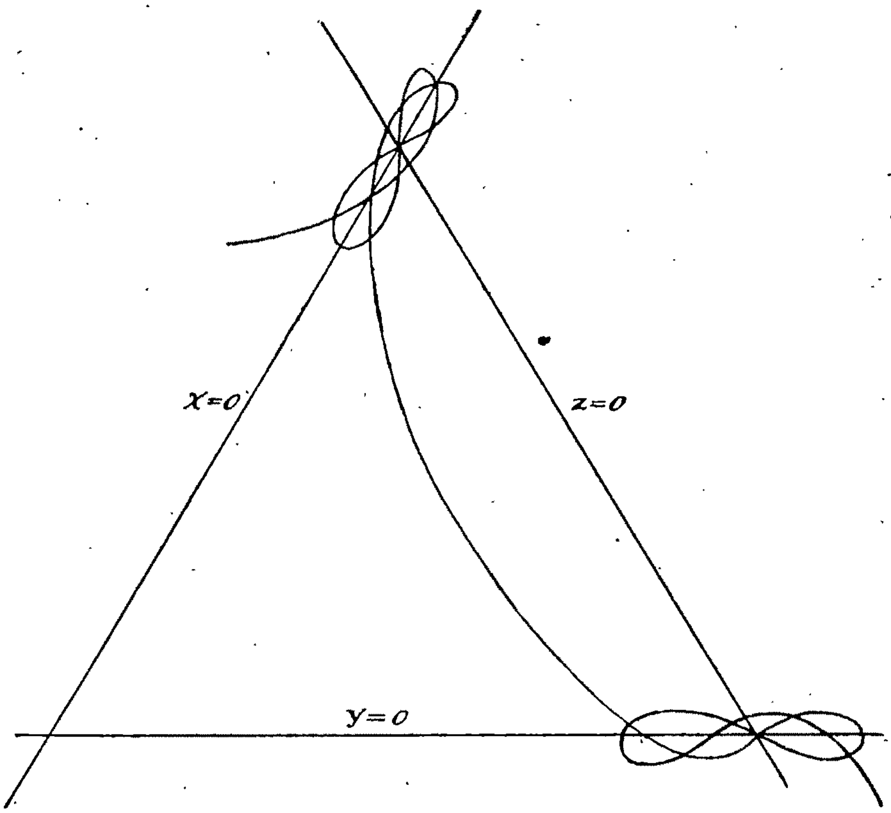


FIG. 22.

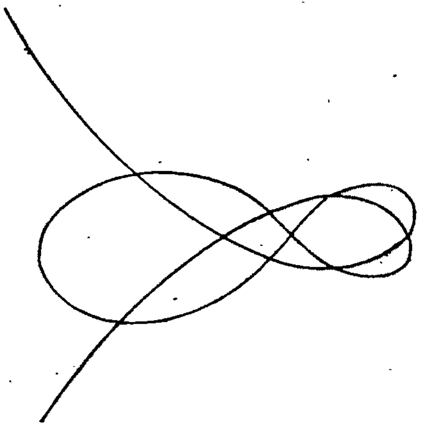


FIG. 18.

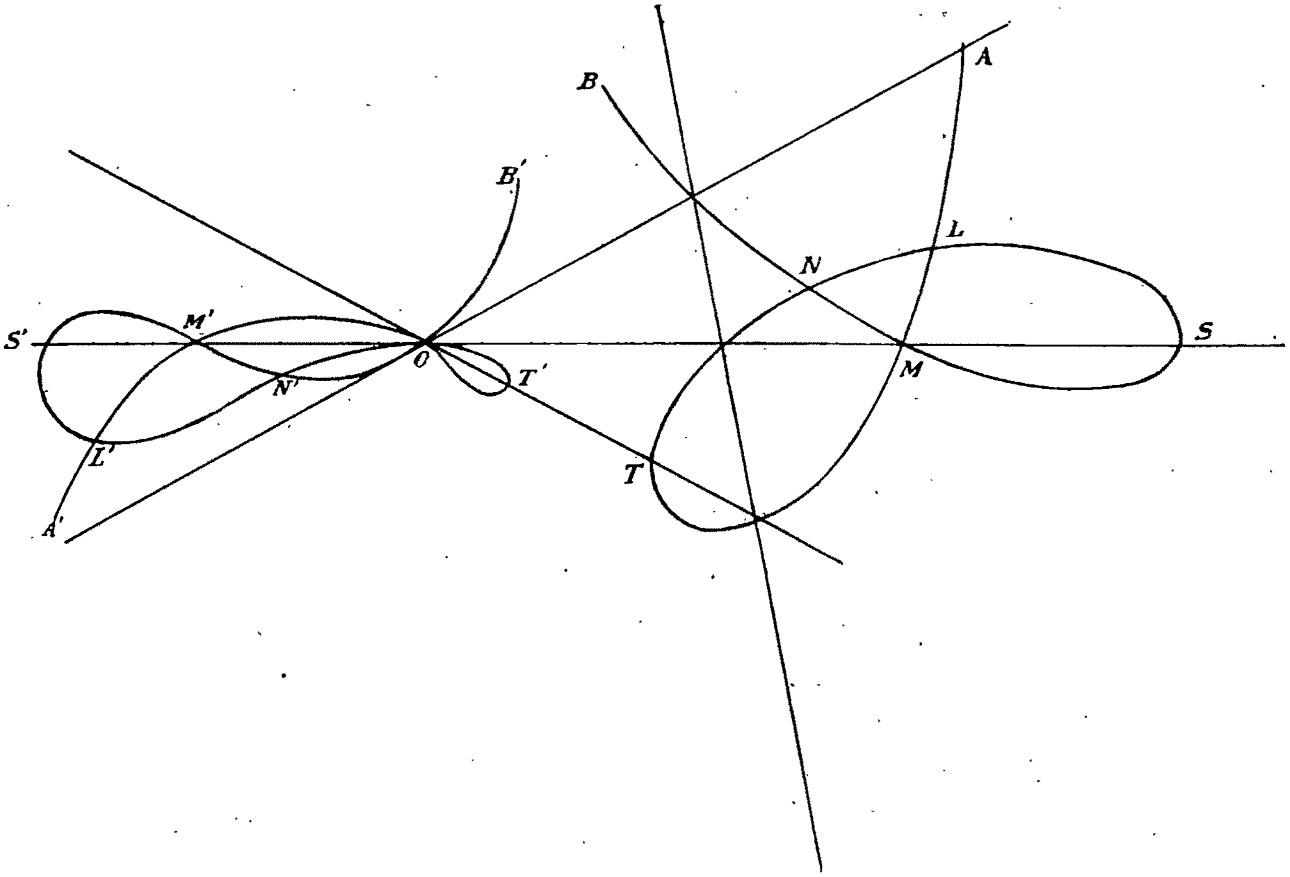


FIG. 19.

17. As a further example, take

$$y^6 - 2x^4y^3 + x^8 - 9x^5y^4 = 0 \tag{1}$$

a curve having at the origin a singularity affording the expansions of Professor Cayley's paper in Vol. VII of the Quarterly Journal, and giving $\delta = 16$, $\varkappa = 5$.

The inverses are

$$y_1^6 - 2x_1y_1^3 + x_1^2 - 9x_1^3y_1^4 = 0 \tag{2}$$

$$y_2^4 - 2x_2y_2^2 + x_2^2 - 9x_2^3y_2^5 = 0 \tag{3}$$

$$y_3^2 - 2x_3y_3 + x_3^2 - 9x_3^3y_3^6 = 0 \tag{4}$$

In this, write $x_3 - y_3 = u_3$, and it becomes

$$u_3^2 - 9y_3^6(u_3 + y_3)^3 = 0.$$

It is convenient to start afresh at this point and analyse independently the singularity

$$y^2 - 9x^6(x + y)^3 = 0 \tag{5}$$

We get

$$y_1^2 - 9x_1^7(1 + y_1)^3 = 0 \tag{6}$$

$$y_2^2 - 9x_2^5(1 + x_2y_2)^3 = 0 \tag{7}$$

$$y_3^2 - 9x_3^3(1 + x_3^2y_3)^3 = 0 \tag{8}$$

$$y_4^2 - 9x_4(1 + x_4^3y_4)^3 = 0 \tag{9}$$

giving Fig. (23), which shows the nature of the singularity on (5).

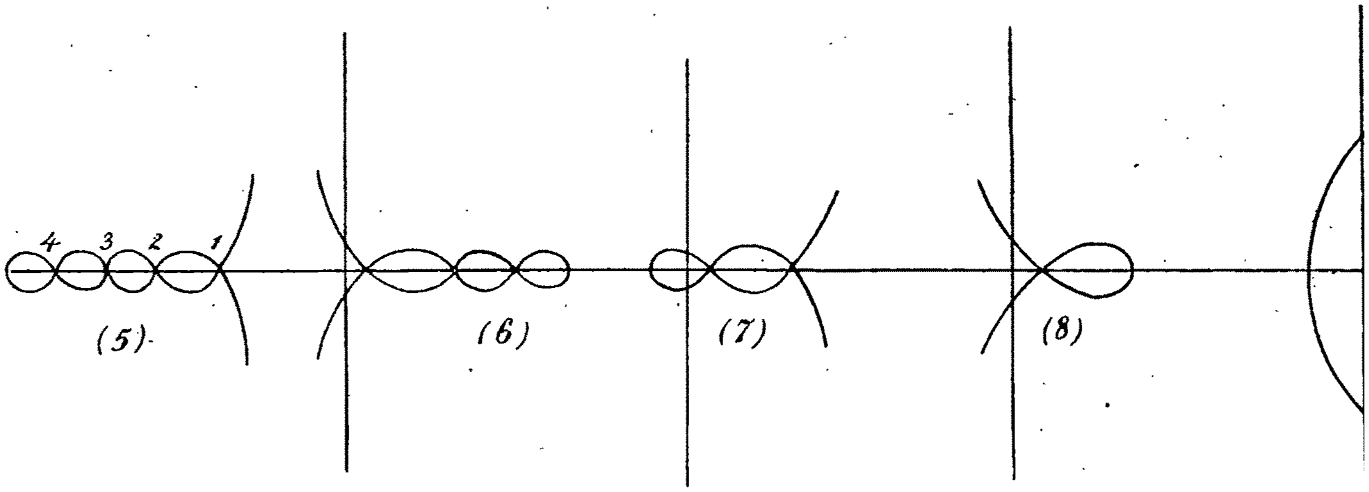


FIG. 23.

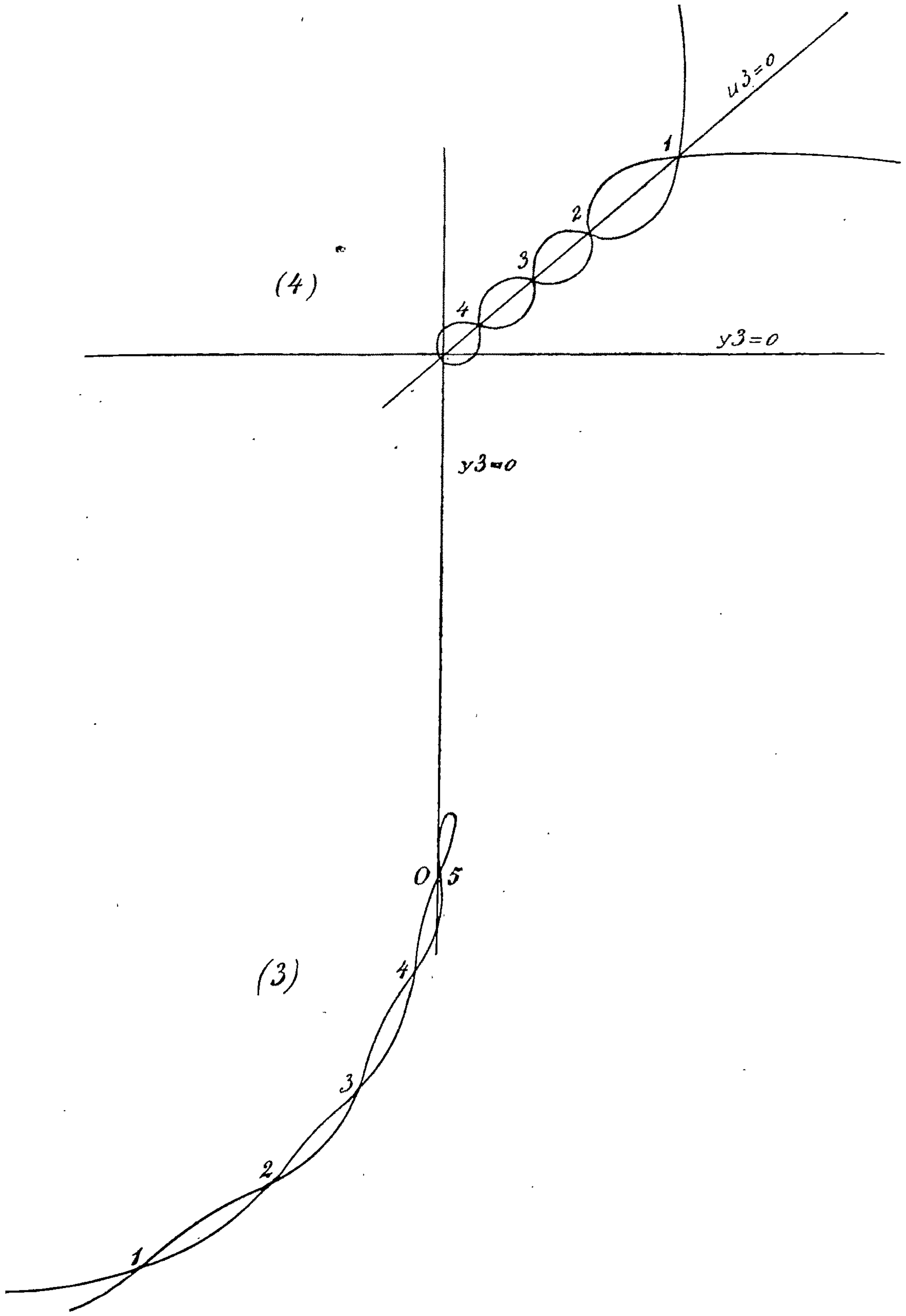


FIG. 24.

The transition from (5) (i. e. (4)) to (3) is shown in Fig. (24), the straight line on which lie the four nodes being now curved into a conic.*

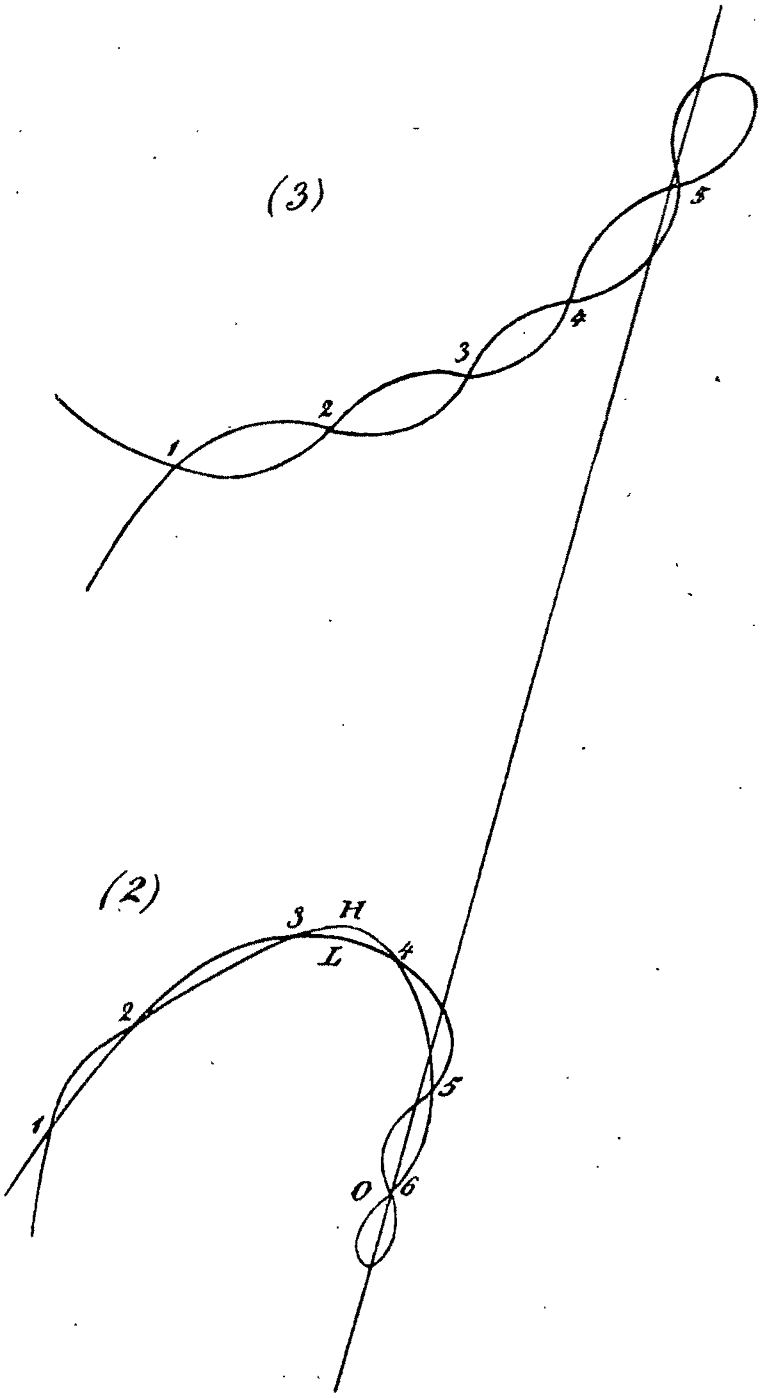


FIG. 25.

---

* In Figs. (23) to (26), for the sake of distinctness, the tangents are more widely separated than is justified by the construction; and the two branches that cross and recross at 1 . 2 . 3 . etc. are drawn with inflexions, as the scale is too small to keep them distinct in any other way.

The line $x_2 = 0$ (i. e. $x_3 = 0$) meets (3) in *four* points; the base line meets it in *two*. The transition from (3) to (2) introduces therefore one more node, so the first inverse of the given curve has six dps, consecutive points on some curve; one of the dps being a cusp (Fig. (25)).

In passing from (2) to (1) the base line is $x_1 = 0$, meeting the curve in six points; this finally introduces the sextic point, as in Fig. (26).

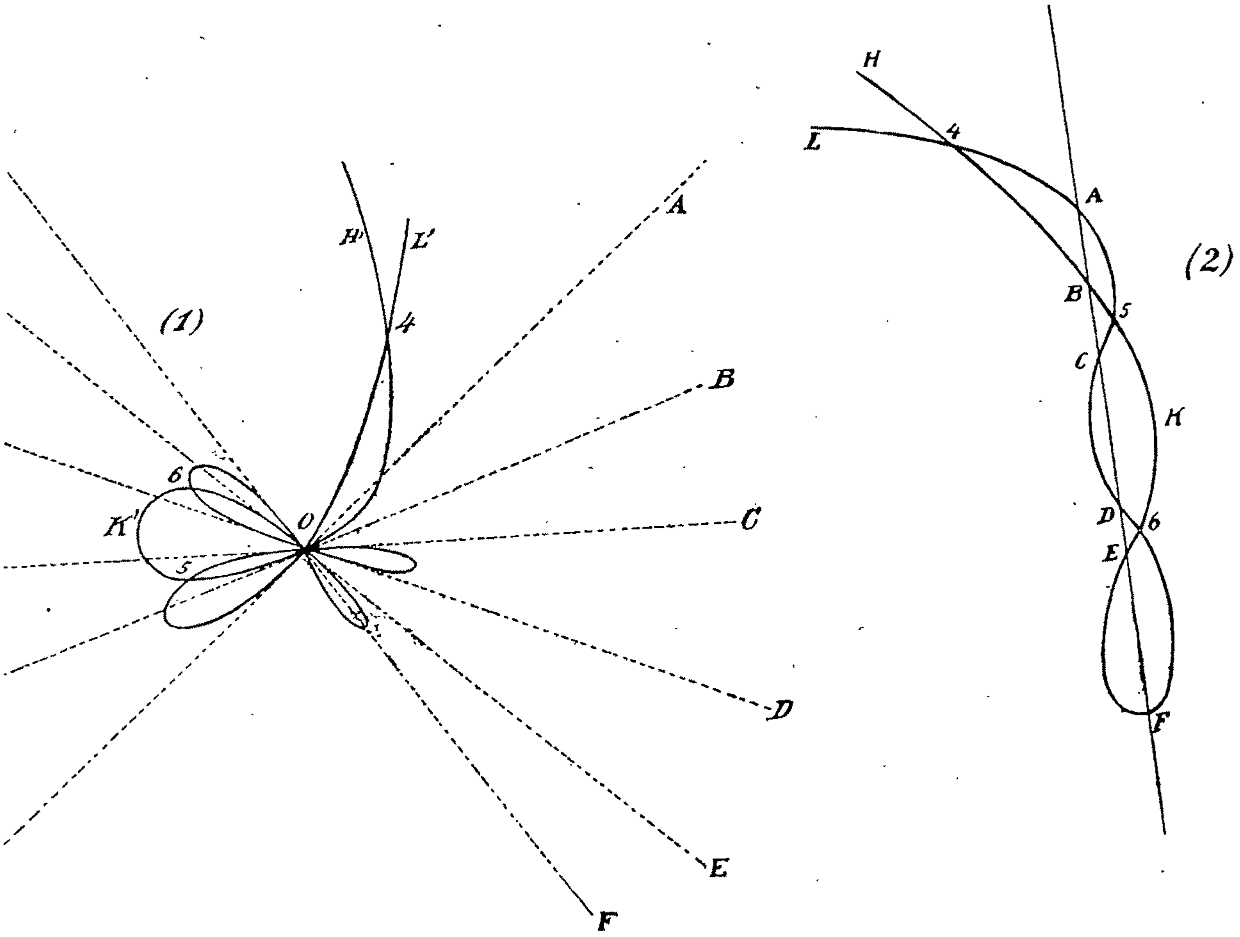


FIG. 26.

The sextic point giving 15 nodes, with 5 loops, we have

$$\delta + \varkappa = 15 + 6 = 21, \quad \varkappa = 5,$$

the numbers given in the paper referred to. Displacing slightly the components of the sextic point, Fig. (27), the course of the curve is made more distinct. The appearance presented is that of a ramphoid cusp.

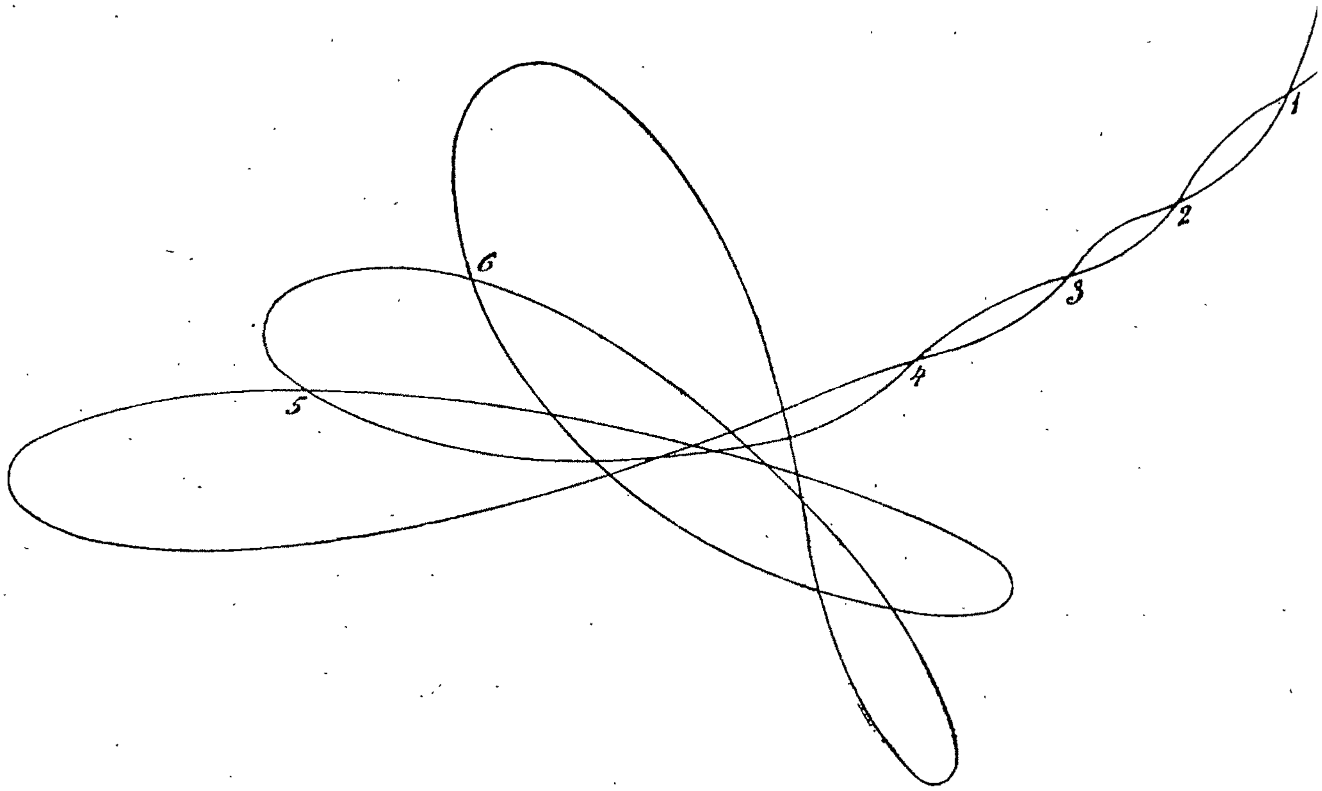


FIG. 27.

18. Applying the process to $y^m = x^p$ (p prime to m and $> m$) (Halphen, "Mémoire sur les points singuliers des courbes algébriques planes," Article II) the first inverse is

$$y_1^m = x_1^{p-m},$$

the second is

$$y_2^m = x_2^{p-2m},$$

and so on, till the exponent of x is less than the exponent of y.

Let

$$\left.\begin{aligned} p &= mk_1 + q, \\ m &= qk_2 + r, \\ q &= rk_3 + s, \text{ etc.} \end{aligned}\right\} \qquad (1)$$

Reducing to the form $y^m = x^q$, $q < m$, we see that the base is now the tangent, and meets the curve (the k_1^{th} inverse) in m points. There are therefore in the original $m-1$ cusps, i. e. $\varkappa = m-1$. Further, we have resolved the point into k_1 consecutive m-points, together with the dps that are given by

$$y^m = x^q \ (q < m).$$

We have then, if $\underset{m,\,p}{\delta}$ denotes the number of nodes in the singularity $y^m = x^p (p > m)$,

$$\underset{m,\,p}{\delta} = k_1 \times \frac{m(m-1)}{2} + \underset{q,\,m}{\delta}.$$

This gives us

$$\begin{aligned} \underset{m,\,p}{\delta} &= k_1\,\frac{m(m-1)}{2} + k_2\,\frac{q(q-1)}{2} + k_3\,\frac{r(r-1)}{2} + \dots . \\ &= \tfrac{1}{2}\{(p-q)(m-1) + (m-r)(q-1) + (q-s)(r-1) + \dots .\} \end{aligned}$$

from formulæ (1), in which the last of the quantities p, m, q, ... is $= 1$.

$$\begin{aligned} \therefore \underset{m,\,p}{\delta} &= \tfrac{1}{2}\left\{\begin{array}{l} (p-q)m + (m-r)q + (q-s)r + \dots . \\ -p + q - m + r - q + s \dots . + 1 \end{array}\right\} \\ &= \tfrac{1}{2}\{pm - p - m + 1\} \\ &= \frac{(p-1)(m-1)}{2}. \end{aligned}$$

We get therefore, for the point-components of the singularity $y^m = x^p$,

$$\varkappa = m - 1,\ \delta + \varkappa = \frac{(p-1)(m-1)}{2} \quad \text{(Halphen.)}$$

As was explained in §8, the process does not ordinarily enable us to count the inflexions, though in certain cases they can be detected by means of the tangents from O. In each inversion used in the present example, the tangent is either the original tangent or the base line; and consequently, there being only one complete branch, the inflexions can be counted. The number given by Brill's formulæ (Math. Ann. XVI, 400), $p - m - 1$, presents itself as follows:

The first inverse is $y_1^m = x_1^{p-m}$.

(i) If now $p - m < m$, so that $x_1 = 0$ is the tangent, this singularity contains $p - m - 1$ cusps; each of these indicates an evanescent loop; O *not* being on the coincident tangents, there is from O, to each of these loops, one tangent; these tangents from O give inflexions on the original, in number $p - m - 1$; and this being the only way in which inflexions can occur in the singularity considered, we have for the total number $p - m - 1$.

(ii) If $p - m > m$, $y_1 = 0$ is the tangent, and meets the inverse in $p - m$ points; there being only one branch, this gives $p - m - 1$ contacts, and therefore in the original $p - m - 1$ inflexions.

Finally, knowing that the equation of the reciprocal of the curve $y^m = x^p$ is of the same form, and that the reciprocal singularity contains $p - m - 1$ cusps, $m - 1$ inflexions, we see that the reciprocal singularity has an equation $u^{p-m} = v^p$; and therefore we have

$$\tau + \iota = \frac{(p-1)(p-m-1)}{2}, \quad \iota = p - m - 1.$$

Thus e. g. $x^5y^3 = z^8$.

The 3 . 8 point gives $\delta + \varkappa = 7, \quad \varkappa = 2,$

$\iota = 4, \quad \tau + \iota = 14,$

i. e. $\delta = 5$, $\varkappa = 2$, $\tau = 10$, $\iota = 4$.

The 5 . 8 point gives $\delta + \varkappa = 14, \quad \varkappa = 4,$

$\iota = 2, \quad \tau + \iota = 7,$

i. e. $\delta = 10$, $\varkappa = 4$, $\tau = 5$, $\iota = 2$.

19. In the case of a singularity with coincident tangents, the danger in separating the tangents is that the neighboring multiple points may be destroyed. But these are indicated by multiple points on the base line; we can therefore separate the tangents, provided that, on the inverse, no multiple points are thereby lost in the immediate neighborhood of the base line; and we can substitute nodes and loops for the cusps, by displacing the base line so that there may no longer be contact.

Take for example the curve (Cramer)

$$(y - x^2)^3 - x^3y^2 = 0, \tag{1}$$

this has two consecutive triple points, involving two cusps. The first inverse is

$$(y_1 - x_1)^3 - x_1^2y_1^2 = 0, \tag{2}$$

which by the substitution $y_1 = x_1 + u_1$, becomes

$$u_1^3 - x_1^2(x_1 + u_1)^2 = 0. \tag{3}$$

The inverse of this, obtained by $x_1 = x_2$, $u_1 = x_2y_2$, is

$$y_2^3 - x_2(1 + y_2)^2 = 0. \tag{4}$$

The curve (4) has an inflexion at x_2y_2; no multiple points are lost if we resolve the inflexion in the ordinary way, separating the three consecutive points by writing $y_2(y_2-\lambda)(y_2+\lambda)$ instead of y_2^3. We get then, instead of (4),

$$(y_2^2-\lambda^2)y_2-x_2(1+y_2)^2=0\,; \tag{4'}$$

instead of (3) we have

$$(u_1^2-\lambda^2x_1^2)u_1-x_1^2(x_1+u_1)^2=0\,; \tag{3'}$$

instead of (2), $\{(y_1-x_1)^2-\lambda^2x_1^2\}(y_1-x_1)-x_1^2y_1^2=0\,.$ (2')

In passing to (1), $x_1=0$ is the base line, and it passes through the triple point at x_1y_1; take instead as base line $x_1=\mu$; i. e. change the x_1 into $x_1+\mu$; and we get

$$\{(y_1-x_1-\mu)^2-\lambda^2(x_1+\mu)^2\}\{y_1-x_1-\mu\}-\{x_1+\mu\}^2y_1^2=0,$$

which gives us, instead of (1),

$$\{(y-x^2-\mu x)^2-\lambda^2x^2(x+\mu)^2\}\{y-x^2-\mu x\}-xy^2(x+\mu)^2=0\,. \tag{1'}$$

This has the two triple points separated, they being now at $0\,.\,0$ and at $-\mu\,.\,0$. The tangents at each are separated, and the cusps represented by loops.

If however we take $y^3=x^5$, which gives as the first inverse $y_1^3=x_1^2$, we cannot separate the *three* points on the base line; for by so doing we should destroy a dp. But we may substitute for y_1^3, $y_1^2(y_1+\lambda)$, which reduces the cusp to a node and loop; and then take a new base line which does not pass through the node, as in the last example.

BRYN MAWR COLLEGE, Bryn Mawr, Pa., June, 1892.

On the Roots of Matrices.

By W. H. Metzler.

CONTENTS.

§1. Representation of a matrix.
  I. Latent roots $\neq 0$.
  (a). All roots distinct.
    1. Difference equation and its solution.
    2. Powers of φ.
    3. A_0, B_0, etc., idempotent and mutually nilfactorial.
    4. Rational function of φ.
    5. Identical equation to φ.
    6. Sylvester's Formula.
  (b). Sets of equal latent roots.
    7. Difference equation and its solution.
    8. Powers of φ.
    9. Rational function of φ.
    10. Vacuity and nullity.
    11. Identical equation.

  II. Some latent roots zero.
    12. Difference equation and its solution.
    13. Expression for unity.
    14. Powers of φ.
    15. Rational integral function of φ.
    16. Identical equation to φ.

§2. Properties of the A's, B's, etc.
    17. Results already obtained.
    18. Relation between the A's.
    19. Relation between the N's.

§3. 20. Law of latency.

§4. 21. Nullity of the factors of the identical equation.

§5. Roots of a matrix.
- I. Latent roots $\neq 0$.
  - 22. m^{th} roots of φ.
  - 23. Number of distinct m^{th} roots of φ.
  - 24. Examples.
  - 25. Negative fractional indices.
- II. Some latent roots zero.
  - 26. All latent roots zero—roots of zero.
  - 27. There is always a q^{th} root of a nilpotent matrix unless the law of nullity prohibits.
  - 28. All possible types of roots of nilpotent matrices of orders 3–10 inclusive.
  - 29. Some latent roots zero and others different from zero.
  - 30. There is always a q^{th} root of φ unless the law of nullity prohibits.

§6. 31. Transcendental functions of a matrix.
(a). Exponential. (b). Logarithmic. (c). Sine. (d). Sin^{-1}.

Introduction.

In his memoir on Matrices (Phil. Trans. 1858) Prof. Cayley enunciated the theorem: "The determinant, having for its matrix a given matrix less the same matrix considered as a single quantity involving the matrix unity, is equal to zero." The equation implied in this theorem is known as Cayley's "identical equation." Subsequently (in the Mess. Math. Vol. XIII, p. 139), Mr. A. R. Forsyth gave a proof of this identical equation for matrices of the third order, based upon the solution of a system of linear difference equations.* Forsyth's method is applicable to matrices of any order. Considerable simplicity is gained, however, by the employment of non-scalar equations instead of the scalar equations employed by Forsyth.

I have employed this modification of Forsyth's method to prove Sylvester's law of latency and Sylvester's theorem. In addition I have by this method investigated the existence of roots of matrices for different indices and in particular the roots of nilpotent matrices.

For valuable suggestions in the working of this paper I am indebted to Dr. Henry Taber.

---

*Sylvester stated (in the Johns Hopkins Univ. Circ. No. 28, 1884) that a proof of the identical equation could be obtained by the method of linear difference equations.

§1.—*Representation of a Matrix.*

In obtaining the representation of matrices by the method of difference equations, I shall make two general divisions: I, in which all latent roots are different from zero, and II, in which some latent roots are zero.

I. *Latent roots* $\neq 0$.—When the latent roots are different from zero, it will be found convenient to distinguish between two cases (a), in which all the latent roots are distinct, and (b), in which there are groups or sets of equal latent roots.

(a). *All latent roots distinct.*

1. The method readily presents itself on considering a few examples, as follows:

1).—*Matrix of order* 2.

Suppose $\phi = \begin{Bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{Bmatrix}$, where ϕ_{11}, ϕ_{12}, etc., represent the constituents in the positions indicated by their suffices.

Define integer powers of ϕ by $\phi^{n+1} = \phi \cdot \phi^n$; and let

$$\phi^n = \begin{Bmatrix} (\phi^n)_{11} & (\phi^n)_{12} \\ (\phi^n)_{21} & (\phi^n)_{22} \end{Bmatrix}.$$

We have then

$$\begin{Bmatrix} (\phi^{n+1})_{11} & (\phi^{n+1})_{12} \\ (\phi^{n+1})_{21} & (\phi^{n+1})_{22} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{Bmatrix} \begin{Bmatrix} (\phi^n)_{11} & (\phi^n)_{12} \\ (\phi^n)_{21} & (\phi^n)_{22} \end{Bmatrix}.$$

Expanding the right-hand member we get the equations

$$\begin{aligned} (\phi^{n+1})_{11} &= \phi_{11}(\phi^n)_{11} + \phi_{12}(\phi^n)_{21}, \\ (\phi^{n+1})_{21} &= \phi_{21}(\phi^n)_{11} + \phi_{22}(\phi^n)_{21}, \\ &\text{etc., etc.} \end{aligned}$$

The left-hand member of the first equation is $E(\phi^n)_{11}$, and of the second equation is $E(\phi^n)_{21}$, where E is the enlargement symbol of finite differences.

If we multiply the first equation by ϕ_{22} and subtract from the product the second equation multiplied by ϕ_{12}, we get

$$\{E^2 - (\phi_{11} + \phi_{22})E + \phi_{11}\phi_{22} - \phi_{12}\phi_{21}\}(\phi^n)_{11} = 0,$$

or $(E - g_1)(E - g_2)(\phi^n)_{11} = 0$, where g_1 and g_2 are the roots of the equation

$$\begin{vmatrix} \phi_{11} - x & \phi_{12} \\ \phi_{21} & \phi_{22} - x \end{vmatrix} = 0.$$

The function $\begin{vmatrix} \phi_{11}-x & \phi_{12} \\ \phi_{21} & \phi_{22}-x \end{vmatrix}$ is called the *latent function* of ϕ;* g_1 and g_2 are called the *latent roots* of ϕ.

Similarly we obtain

$$(E-g_1)(E-g_2)(\phi^n)_{12}=0,$$

and generally

$$(E-g_1)(E-g_2)(\phi^n)_{rs}=0, \qquad \left.\begin{matrix} r \\ s \end{matrix}\right\}=1,\ 2.$$

The solution of this difference equation is

$$(\phi^n)_{rs}=A_{rs}g_1^n+B_{rs}g_2^n,$$

where A_{rs} and B_{rs} are constants determined by giving n successively the values 0 and 1.

$$\therefore\ (1)_{rs}\dagger=A_{rs}+B_{rs},$$

$$\phi_{rs}=A_{rs}g_1+B_{rs}g_2;$$

$$\therefore\ A_{rs}=\frac{\begin{vmatrix} (1)_{rs} & 1 \\ \phi_{rs} & g_2 \end{vmatrix}}{\Delta},\ B_{rs}=\frac{\begin{vmatrix} 1 & (1)_{rs} \\ g_1 & \phi_{rs} \end{vmatrix}}{\Delta} \text{ and } \Delta=\begin{vmatrix} 1 & 1 \\ g_1 & g_2 \end{vmatrix}.$$

We may write instead of the above equation,

$$\phi=A_0g_1+B_0g_2,\ \text{where } A_0=\frac{\begin{vmatrix} 1 & 1 \\ \phi & g_2 \end{vmatrix}}{\Delta}$$

$$\text{and } B_0=\frac{\begin{vmatrix} 1 & 1 \\ g_1 & \phi \end{vmatrix}}{\Delta};$$

(i. e., we may substitute for the scalar difference equations non-scalar ones), since

$$\phi^n=\left\{\begin{matrix} \dfrac{g_1^n(g_2-\phi_{11})}{\Delta}+\dfrac{g_2^n(\phi_{11}-g_1)}{\Delta} & \dfrac{g_1^n(-\phi_{12})+g_2^n\phi_{12}}{\Delta} \\ \dfrac{g_1^n(-\phi_{21})+g_2^n\phi_{21}}{\Delta} & \dfrac{g_1^n(g_2-\phi_{12})+g_1^n(\phi_{22}-g_1)}{\Delta} \end{matrix}\right|$$

$$=\frac{g_1^n(g_2-\phi)}{\Delta}+\frac{g_2^n(\phi-g_1)}{\Delta}$$

$$=A_0g_1^n+B_0g_2^n, \text{ which is the solution of } (E-g_1)(E-g_2)\phi^n=0.$$

Similarly it may be shown for matrices of any order.

---

* In general of $\phi=\left\{\begin{matrix} \phi_{11}\phi_{12}\ldots\ldots\phi_{1\omega} \\ \phi_{21}\phi_{22}\ldots\ldots\phi_{2\omega} \\ \vdots \qquad\qquad \vdots \\ \phi_{\omega 1}\ldots\ldots\ldots\phi_{\omega\omega} \end{matrix}\right|$, then $\begin{vmatrix} \phi_{11}-x & \phi_{12}\ldots\ldots & \phi_{1\omega} \\ \phi_{21} & \phi_{22}-x\ldots & \phi_{2\omega} \\ \vdots & \vdots & \vdots \\ \phi_{\omega 1}\ldots & \ldots\ldots & \phi_{\omega\omega}-x \end{vmatrix}$ is the latent function of ϕ.

† $(1)_{rs}=\begin{cases} 0 \text{ for } r\neq s \\ 1\ \text{``}\ r=s \end{cases}$.

2).—*Matrix of order* 3.

Suppose $\phi = \begin{vmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix}$ and $\phi^n = \begin{vmatrix} (\phi^n)_{11} & (\phi^n)_{12} & (\phi^n)_{13} \\ (\phi^n)_{21} & (\phi^n)_{22} & (\phi^n)_{23} \\ (\phi^n)_{31} & (\phi^n)_{32} & (\phi^n)_{33} \end{vmatrix}$.

Proceeding as in the previous case and forming the difference equation, we get $(E-g_1)(E-g_2)(E-g_3)\phi^n = 0$, where g_1, g_2, g_3 are the latent roots of ϕ.

The solution of this is

$$\phi^n = A_0 g_1^n + B_0 g_2^n + C_0 g_3^n,$$

$$\text{and } A_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ \phi & g_2 & g_3 \\ \phi^2 & g_2^2 & g_3^2 \end{vmatrix}}{\Delta}, \quad B_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ g_1 & \phi & g_3 \\ g_1^2 & \phi^2 & g_3^2 \end{vmatrix}}{\Delta}$$

$$C_0 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ g_1 & g_2 & \phi \\ g_1^2 & g_2^2 & \phi^2 \end{vmatrix}}{\Delta}, \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ g_1 & g_2 & g_3 \\ g_1^2 & g_2^2 & g_3^2 \end{vmatrix}$$

3).—*Matrix of order* ω.

Suppose ϕ has as latent roots $g_1, g_2, g_3, \ldots . g_\omega$.

Forming the difference equation we get

$$(E-g_1)(E-g_2)\ldots.(E-g_\omega)\phi^n = 0,$$

the solution of which is

$$\phi^n = A_0 g_1^n + B_0 g_2^n + \ldots . + W_0 g_\omega^n.$$

Giving n successively the ω different values $0, 1, \ldots, \omega-1$, we get ω equations linear in A_0, B_0, etc., which are sufficient to determine A_0, B_0, etc., as follows:

$$A_0 = \frac{\begin{vmatrix} 1 & 1 & \ldots & 1 \\ \phi & g_2 & \ldots & g_\omega \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ \phi^{\omega-1} & g_2^{\omega-1} & \ldots & g_\omega^{\omega-1} \end{vmatrix}}{\Delta}, \text{ and similarly, for } B_0,\ C_0, \text{ etc.,}$$

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ g_1 & g_2 & \ldots & g_\omega \\ \vdots & \vdots & \ldots & \vdots \\ \vdots & \vdots & \ldots & \vdots \\ g_1^{\omega-1} & g_2^{\omega-1} & \ldots & g_\omega^{\omega-1} \end{vmatrix}.$$

2. *Powers of ϕ.*—The matrix all of whose constituents are zero except those along the principal diagonal, which are units, is called the *matrix unity* and is generally denoted by 1.

If from the array of constituents representing a matrix ϕ we form another matrix Φ by replacing each constituent of the first array by the logarithmic differential derivative with respect to that constituent of the determinant of the array; then the product of ϕ and the transverse of Φ in either order is the matrix unity.

The transverse of Φ, written $\breve{\Phi}$, is denoted by ϕ^{-1} and is termed the *reciprocal* of ϕ.

From this definition of the reciprocal, which is as given by Sylvester (Am. Jour., Vol. VI), we see that a matrix has no reciprocal when any of its latent roots are zero.

We may observe that the expressions for A_0, B_0, etc., as given in the preceding examples, are functions of ϕ containing the $(\omega-1)^{\text{th}}$ and lower powers; and consequently we have a formula for expressing all other powers of ϕ in terms of these.

3. *A_0, B_0, etc., are idempotent and mutually nilfactorial.*—The formula for ϕ^n gives

$$\phi^\omega = A_0 g_1^\omega + B_0 g_2^\omega + \dots + W_0 g_\omega^\omega,$$

$$\text{or } \phi^\omega = \frac{(\phi-g_2)(\phi-g_3)\dots(\phi-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots(g_1-g_\omega)}g_1^\omega + \frac{(\phi-g_1)(\phi-g_3)\dots(\phi-g_\omega)}{(g_2-g_1)(g_3-g_3)\dots(g_2-g_\omega)}g_2^\omega + \dots$$

$$+\frac{(\phi-g_1)(\phi-g_2)\dots(\phi-g_{\omega-1})}{(g_\omega-g_1)(g_\omega-g_2)\dots(g_\omega-g_{\omega-1})}g_\omega^\omega,$$

$$\text{since } A_0 = \frac{(\phi-g_2)(\phi-g_3)\dots(\phi-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots(g_1-g_\omega)},$$

$$B_0 = \frac{(\phi-g_1)(\phi-g_3)\dots(\phi-g_\omega)}{(g_2-g_1)(g_2-g_3)\dots(g_2-g_\omega)}, \text{ etc.}$$

Writing a scalar symbol x for ϕ in the above formula we have

$$x^\omega = \frac{(x-g_2)(x-g_3)\dots(x-g_\omega)}{(g_1-g_2)(g_1-g_3)\dots(g_1-g_\omega)}g_1^\omega + \frac{(x-g_1)(x-g_3)\dots(x-g_\omega)}{(g_2-g_1)(g_2-g_3)\dots(g_2-g_\omega)}g_2^\omega + \text{etc.}\dots,$$

an equation in x of the degree ω, whose roots are $g_1, g_2 \dots g_\omega$, as is evident on substituting g_1 for x, g_2 for x, etc., in the equation. The equation may then be written as follows:

$$(x - g_1)(x - g_2) \dots (x - g_\omega) = 0, \text{ or replacing } x \text{ by } \phi \text{ we have}$$
$$(\phi - g_1)(\phi - g_2) \dots (\phi - g_\omega) = 0.$$

Again the formula for ϕ^n gives

$$1 = A_0 + B_0 + \dots + W_0;$$
$$\therefore \quad A_0 = A_0^2,$$
$$B_0 = B_0^2, \text{ etc.},$$

since A_0B_0, A_0C_0, etc., contain all the factors of the above equation and therefore vanish.

This proves that the letters A_0, B_0, etc., are idempotent and mutually nilfactorial.

4. *Rational function of ϕ.*—Having obtained expressions for powers of ϕ, we come naturally to the consideration of a rational integral function of ϕ of an order not less than ω, which may be written as follows:

$$\overset{n}{\underset{0}{\sum}}{}^{\lambda} a_\lambda \phi^\lambda = \overset{n}{\underset{0}{\sum}}{}^{\lambda} a_\lambda [A_0 g_1^\lambda + B_0 g_2^\lambda + \dots + W_0 g_\omega^\lambda],$$

or
$$F\phi = A_0 F g_1 + B_0 F g_2 + \dots + W_0 F g_\omega.$$

We saw that a matrix ϕ had a reciprocal provided none of its latent roots were zero, and, since a rational integral function of a matrix is a matrix, the rational integral function $F\phi$ will have a reciprocal $(F\phi)^{-1}$ provided none of its latent roots are zero.

The reciprocal of $F\phi$ would be written

$$(F\phi)^{-1} = A_0(Fg_1)^{-1} + B_0(Fg_2)^{-1} + \dots + W_0(Fg_\omega)^{-1};$$

because
$$F\phi \,.\, (F\phi)^{-1} = A_0 + B_0 + \dots + W_0$$
$$= 1.$$

We may write then a rational function of ϕ as follows:

$$f\phi = \frac{F_1\phi}{F_2\phi} = \frac{A_0F_1g_1 + B_0F_1g_2 + \dots + W_0F_1g_\omega}{A_0F_2g_1 + B_0F_2g_2 + \dots + W_0F_2g_\omega},$$

where $F_1\phi$ and $F_2\phi$ are rational integral functions of ϕ and where none of the latent roots of $F_2\phi$ are zero. This function may be written

$$f\phi = A_0\frac{F_1g_1}{F_2g_1} + B_0\frac{F_1g_2}{F_2g_2} + \ldots\ldots + W_0\frac{F_1g_\omega}{F_2g_\omega},$$
$$\text{or } f\phi = A_0fg_1 + B_0fg_2 + \ldots\ldots + W_0fg_\omega.$$

5. *Identical equation.*—The formula for ϕ^n gives

$$\phi^\omega = A_0g_1^\omega + B_0g_2^\omega + \ldots\ldots + W_0g_\omega^\omega,$$

and, as we have already observed, A_0, B_0, etc., are functions of the first $(\omega - 1)$ powers of ϕ and unity, and consequently the above is an equation between the first ω powers of ϕ and unity. No other equation of this or lower order will be satisfied by ϕ, since A_0, B_0, etc., are linearly independent and therefore this is Cayley's "identical equation."

The A_0, B_0, etc., may readily be shown to be linearly independent; for if they are not, suppose the relation

$$a_0A_0 + b_0B_0 + \ldots\ldots + w_0W_0 = 0, \text{ where } a_0, b_0 \ldots\ldots w_0$$

are scalar constants. Multiplying this by A_0 we get

$$a_0A_0 = 0,$$

since A_0 is idempotent and nilfactorial with respect to all the other letters B_0, C_0, etc., as shown in Art. 3; therefore $a_0 = 0$.

Similarly we may show that $b_0 = c_0 = \ldots\ldots = w_0 = 0$ and there is therefore no linear relation between these letters.

Again we have

$$\begin{aligned}
\phi &= A_0g_1 + Bg_2 + \ldots\ldots + W_0g_\omega, \\
\phi - g_1 &= B_0(g_2 - g_1) + C_0(g_3 - g_1) + \ldots\ldots + W_0(g_\omega - g_1), \\
\phi - g_2 &= A_0(g_1 - g_2) + C_0(g_3 - g_2) + \ldots\ldots + W_0(g_\omega - g_2), \\
&\vdots \\
\phi - g_\omega &= A_0(g_1 - g_\omega) + B_0(g_2 - g_\omega) + \ldots\ldots + V_0(g_{\omega-1} - g_\omega).
\end{aligned}$$

Then since A_0, B_0, etc., are mutually nilfactorial we have

$$(\phi - g_1)(\phi - g_2) \ldots\ldots (\phi - g_\omega) = 0,$$

which is the identical equation in product form. Here again it is obvious that ϕ satisfies no other equation of this or lower order, since the letters A_0, B_0, etc., are linearly independent.

Hereafter when I use the term "the letters" without further specification, I shall mean the A's, B's, etc.

6. *Sylvester's Formula.*—The rational function of Art. 4, when written in the form

$$f\phi = A_0 f g_1 + B_0 f g_2 + \ldots\ldots + W_0 f g_\omega,$$

is obviously Sylvester's formula for the particular case of a rational function, since the expression for A_0 is $\dfrac{(\phi - g_2)(\phi - g_3)\ldots\ldots(\phi - g_\omega)}{(g_1 - g_2)(g_1 - g_3)\ldots\ldots(g_1 - g_\omega)}$ and similarly for the other letters.

We have then, thus far, a means of reducing a rational function of ϕ to a rational integral function containing only the first ω powers of ϕ, beginning with $\phi^0 = 1$.

(b). *Sets of equal roots.*

7. Let us for convenience take the three examples that we considered in (a), where now we suppose some of the roots to become equal.

1). *Matrix of order* 2.—Suppose in this case $g_1 = g_2$, then proceeding as in Art. 1, we find for the difference equation

$$(E - g_1)^2 \phi^n = 0,$$

the solution of which is

$$\phi^n = (A_0 + nA_1)\, g_1^n.$$

If $n = 0$, then $1 = A_0$,
" $n = 1$, " $\phi = (A_0 + A_1)\, g_1$;

$$\therefore\quad A_1 = \frac{\phi - g_1}{\Delta}, \text{ and } \Delta = g_1.$$

2). *Matrix of order* 3.—In example 2) of Art. 1, put $g_3 = g_1$, then the difference equation becomes

$$(E - g_1)^2 (E - g_2)\, \phi^n = 0.$$

The solution of this is

$$\phi^n = (A_0 + nA_1)\, g_1^n + B_0 g_2^n.$$

The expressions for the A's and B_0 may be found as before.

3). *Matrix of order* ω.—Suppose we have a matrix of order ω having as latent roots $g_1, g_2 \ldots\ldots g_r, g_s$ of multiplicities $p_1, p_2 \ldots\ldots p_r, p_s$, respectively. The difference equation then becomes

$$(E - g_1)^{p_1}(E - g_2)^{p_2}\ldots\ldots(E - g_s)^{p_s}\phi^n = 0;$$

$$\begin{aligned}\therefore\quad \phi^n &= (A_0 + nA_1 + n^2A_2 + \ldots\ldots + n^{p_1-1}A_{p_1-1})\, g_1^n + (B_0 + nB_1 + \ldots\ldots \\ &\quad + n^{p_2-1}B_{p_2-1})\, g_2^n + \ldots\ldots + (S_0 + nS_1 + \ldots\ldots + n^{p_s-1}S_{p_s-1})\, g_s^n \\ &= g_1^n \overset{p_1-1}{\underset{0}{\sum\nolimits_\lambda}} n^\lambda A_\lambda + g_2^n \overset{p_2-1}{\underset{0}{\sum\nolimits_\lambda}} n^\lambda B_\lambda + \ldots\ldots + g_s^n \overset{p_s-1}{\underset{0}{\sum\nolimits_\lambda}} n^\lambda S_\lambda.\end{aligned}$$

The expressions for the A's, B's, etc., are obtained as before by giving n, ω different values, when we get ω linear equations for their determination. We shall find that

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & \dots\, 1 & \dots\, 1 & 0 & \dots & 0 \\ g_1 & g_1 & g_1 & \dots\, g_2 & \dots\, g_s & g_s & \dots & g_s \\ g_1^2 & 2g_1^2 & 4g_1^2 & \dots\, g_2^2 & \dots\, g_s^2 & 2g_s^2 & \dots & 2^{p_s-1}g_s^2 \\ g_1^3 & 3g_1^3 & 9g_1^3 & \dots\, g_2^3 & \dots\, g_s^3 & 3g_s^3 & \dots & 3^{p_s-1}g_s^3 \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \dots & \cdot & \dots & \cdot \\ g_1^{\omega-1} & (\omega-1)\,g_1^{\omega-1} & (\omega-1)^2 g_1^{\omega-1} & \dots\, g_2^{\omega-1} & \dots\, g_s^{\omega-1} & (\omega-1)g_s^{\omega-1} & \dots & (\omega-1)^{p_s-1}g_s^{\omega-1} \end{vmatrix}$$

The factors of Δ may be found without serious difficulty. They are as follows:

$$\Delta = P \,.\, \prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha} \,.\, \prod_{\beta=1}^{\beta=s} \prod_{\gamma=1}^{\gamma=s} (g_\beta - g_\gamma)^{p_\beta p_\gamma},$$

where $\beta \gtrless \gamma$,

$$P = (p_1-1)!\,(p_1-2)! \dots 2!\,(p_2-1)! \dots (p_s-1)! \dots 2!,$$

and
$$p'_\alpha = (p_\alpha - 1) + (p_\alpha - 2) + \dots + 2 + 1$$
$$= \frac{p_\alpha(p_\alpha - 1)}{2}.$$

Let us for convenience use $\Delta\binom{\phi}{\mu}$ to denote the determinant formed from Δ by substituting for its μ^{th} column the column $\begin{vmatrix} 1 \\ \phi \\ \phi^2 \\ \vdots \\ \phi^{\omega-1} \end{vmatrix}$; also let Σp_λ denote $p_1 + p_2 + \dots + p_\lambda$, where λ has any of the values $1, 2 \dots r$. Then

$$\frac{\Delta\binom{\phi}{1}}{\Delta} = A_0, \quad \frac{\Delta\binom{\phi}{p_1+1}}{\Delta} = B_0, \text{ etc.}$$

We get

$$\Delta\binom{\phi}{1} = P \cdot \prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha} \cdot \frac{\prod\limits_{\beta=1}^{\beta=s} (\phi - g_\beta)^{p_\beta}}{(\phi - g_1)^{p_1}}$$

$$\times F_{p_1-1}(\phi - g_1) \cdot \prod_{\gamma=2}^{\gamma=s} (g_1 - g_\gamma)^{(p_1-1)(p_\gamma-1)} \cdot \prod_{\delta=2}^{\delta=s} \prod_{\epsilon=2}^{\epsilon=s} (g_\delta - g_\epsilon)^{p_\delta p_\epsilon},$$

$$\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta\left({}_{\Sigma p_\lambda}{}^{\phi}{}_{+\,1}\right) = P \cdot \prod_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha} \cdot \frac{\prod\limits_{\beta=1}^{\beta=s} (\phi - g_\beta)^{p_\beta}}{(\phi - g_{\lambda+1})^{p_{\lambda+1}}}$$

$$\times F^{(p_\lambda)}_{p_{\lambda+1}-1}(\phi - g_{\lambda+1}) \cdot \prod_{\gamma=1}^{\gamma=s} (g_{\lambda+1} - g_\gamma)^{(p_{\lambda+1}-1)(p_\gamma-1)} \cdot \prod_{\delta=1}^{\delta=s} \prod_{\epsilon=1}^{\epsilon=s} (g_\delta - g_\epsilon)^{p_\delta p_\epsilon},$$

where $(\lambda + 1) \gtrless \delta \gtrless \varepsilon \gtrless (\lambda + 1) \gtrless \gamma$, and where λ has any of the series of values mentioned before.

The factor $F^{(p_\lambda)}_{p_{\lambda+1}-1}(\phi - g_{\lambda+1})$ is a function of $(\phi - g_{\lambda+1})$ of the order $(p_{\lambda+1} - 1)$, which does not contain as a factor ϕ less either of its latent roots, and is in general different for each of the letters. This formula gives the expression for the numerators of all letters with the subscript zero, and for the numerators of the letters with subscripts other than zero we have

$$\Delta\left({}_{\Sigma p_\lambda}{}^{\phi}{}_{+\,\nu\,+\,1}\right) = P^{(p_\lambda+\nu)} \cdot \frac{\prod\limits_{\alpha=1}^{\alpha=s} g_\alpha^{p'_\alpha}}{g_{\lambda+1}^{p_{\lambda+1}-1}} \cdot \frac{\prod\limits_{\beta=1}^{\beta=s} (\phi - g_\beta)^{p_\beta}}{(\phi - g_{\lambda+1})^{p_{\lambda+1}-\nu}} \cdot F^{(p_\lambda+\nu)}_{p_{\lambda+1}}(\phi - g_{\lambda+1})$$

$$\times \prod_{\gamma=1}^{\gamma=s} (g_{\lambda+1} - g_\gamma)^{(p_{\lambda+1}-1)p_\gamma} \cdot \prod_{\epsilon=1}^{\epsilon=s} (g_\delta - g_\epsilon)^{p_\delta p_\epsilon},$$

where ν has any of the series of values $1, 2, \ldots \overline{p_{\lambda+1} - 1}$, and where

$$\gamma \gtrless (\lambda + 1) \gtrless \delta \gtrless \varepsilon \gtrless (\lambda + 1).$$

The numerical factor remains the same for all the letters with subscript zero, but varies for the others.

Introducing now the denominator Δ on both sides we get

$$A_0 = \prod_{\beta=2}^{\beta=s} (\phi - g_\beta)^{p_\beta} \cdot F_{p_1-1}(\phi - g_1) \div \prod_{\beta=2}^{\beta=s} (g_1 - g_\beta)^{(p_1+p_\beta-1)},$$

$$\vdots \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_i = P^{(i)} \cdot \prod_{\beta=2}^{\beta=s} (\phi - g_\beta)^{p_\beta} \cdot (\phi - g_1)^i \cdot F^{(i)}_{p_1-i-1}(\phi - g_1) \div g_1^{p_1-1} \prod_{\beta=1}^{\beta=s} (g_1 - g_\beta)^{p_\beta} \cdot P;$$

and similarly for the other letters.

8. *Powers of* ϕ.—The difference equation gives an expression for ϕ^n, and (since the latent roots of ϕ are different from zero) n may have any integral value positive, negative, or zero; and here again the A's, B's, etc., are functions of ϕ containing the $(\omega - 1)^{\text{th}}$ and lower powers, so that we have a formula for expressing in terms of these all other powers of ϕ.

9. *Rational function of* ϕ.—Here as in Art. 4 we may obtain any rational integral function of ϕ, and hence any rational function of ϕ, viz. $f\phi = \frac{F_1\phi}{F_2\phi}$, provided none of the latent roots of $F_2\phi$ are zero.

It may be written as follows:

$$f\phi = \sum_0^n {}^{\mu} a_\mu \phi^\mu = \sum_0^n {}^{\mu} a_\mu \left[g_1^\mu \sum_0^{p_1 - 1} {}^{\lambda} \mu^\lambda A_\lambda + g_2^\mu \sum_0^{p_2 - 1} {}^{\lambda} \mu^\lambda B_\lambda + \dots + g_s^\mu \sum_0^{p_s - 1} {}^{\lambda} \mu^\lambda S_\lambda \right].$$

10. *Vacuity and Nullity.*—The determinant of the array of constituents forming the matrix is called the *content* of the matrix, and is denoted by $|\phi|$. If the latent function of ϕ be written in the form

$$x^\omega - m_{\omega-1}x^{\omega-1} + m_{\omega-2}x^{\omega-2} \dots \pm m_1 x \mp m = 0,$$

then it is obvious that m is the content of ϕ; m_1 is the sum of all the principal first minors of $|\phi|$, and generally $m_\varkappa$ is the sum of all the principal $\varkappa^{\text{th}}$ minors of $|\phi|$. If $m = 0$, the matrix ϕ evidently has one latent root zero and is termed *vacuous.* If $m = 0$, and $m_1 \neq 0$, then ϕ has but one latent root zero, and is then said to be simply vacuous or to have the vacuity one. More generally, if all the m's from m to $m_{\varkappa-1}$ are zero, and $m_\varkappa \neq 0$, then ϕ has $\varkappa$ latent roots zero, and is said to have the vacuity $\varkappa$. If $|\phi| \neq 0$, ϕ has no latent root zero, and is then non-vacuous.

If all the $(\varkappa - 1)^{\text{th}}$ minors of the content of a matrix vanish, but not all the $\varkappa^{\text{th}}$ minors, the matrix is said to have a *nullity* $\varkappa$. The nullity may be equal to or less than the vacuity, but never can exceed it.

11. *Identical equation.*—If $g_1, g_2, \dots g_s$ of multiplicities $p_1, p_2, \dots p_s$ respectively are the latent roots of ϕ, then the latent function of ϕ may be written

$$\begin{aligned} F(x) &= (x - g_1)^{p_1}(x - g_2)^{p_2} \dots (x - g_s)^{p_s} \\ &= a_\omega x^\omega + a_{\omega-1}x^{\omega-1} + \dots + a_1 x + a_0 = 0, \end{aligned}$$

where ω is the order of the matrix.

Then
$$\begin{aligned} F(\phi) &= a_\omega\phi^\omega + a_{\omega-1}\phi^{\omega-1} + \dots + a_1\phi + a_0 \\ &= A_0(a_\omega g_1^\omega + a_{\omega-1}g_1^{\omega-1} + \dots + a_1 g_1 + a_0) \\ &+ A_1(\omega a_\omega g_1^\omega + \overline{\omega-1}\,.\,a_{\omega-1}g_1^{\omega-1} + \dots + a_1 g_1) \\ &+ A_2(\omega^2 a_\omega g_1^\omega + \overline{\omega-1}^2\,.\,a_{\omega-1}g_1^{\omega-1} + \dots + a_1 g_1) + \text{etc.} \\ &+ B_0(a_\omega g_2^\omega + a_{\omega-1}g_2^{\omega-1} + \dots + a_1 g_2 + a_0) \\ &+ B_1(\omega a_\omega g_2^\omega + \overline{\omega-1}\,.\,a_{\omega-1}g_2^{\omega-1} + \dots + a_1 g_2) \\ &+ \text{etc., etc.} \\ &= A_0Fg_1 + A_1\left(g_1\frac{d}{dg_1}\right)Fg_1 + \dots + A_i\left(g_1\frac{d}{dg_1}\right)^i Fg_1 + \dots \\ &+ B_0Fg_2 + \dots\dots + B_j\left(g_2\frac{d}{dg_2}\right)^j Fg_2 + \dots \\ &+ \text{etc., etc.} \end{aligned}$$

But
$$\begin{aligned} &Fg_1 = F'g_1 = F''g_1 = \dots = F^{(p_1-1)}g_1 = 0, \\ &Fg_2 = F'g_2 = \dots\dots = F^{(p_2-1)}g_2 = 0, \\ &\text{etc., etc.} \\ \therefore\ &F(\phi) = (\phi - g_1)^{p_1}(\phi - g_2)^{p_2} \dots (\phi - g_s)^{p_s} = 0. \end{aligned}$$

In general ϕ does not satisfy an equation of lower order than the one above, in which case it is the identical equation.

When ϕ does satisfy an equation of lower order the identical equation is said to *degrade*. It is evident that ϕ satisfies but one equation of lowest order.

We have yet to prove, however, that ϕ less either of its latent roots must be contained among the factors of this equation of lowest order. We proceed in the first place to prove certain properties of the A's, B's, etc., which we require.

Having the above equation and knowing the factors of the A's, B's, etc., we see that the letters of one set are nilfactorial with respect to the letters of any other set, and that all the letters except those with subscript zero are nilpotent.

We have
$$\begin{aligned} &1 = A_0 + B_0 + \dots + S_0; \\ \therefore\ &A_0 = A_0^2,\ B_0 = B_0^2, \text{ etc. (multiplying by } A_0, B_0, \text{ etc.)}; \end{aligned}$$
also
$$\begin{aligned} &A_1 = A_0A_1 = A_1A_0,\ A_2 = A_0A_2 = A_2A_0, \text{ etc.}, \\ &B_1 = B_0B_1 = B_1B_0, \text{ etc.}, \\ &\text{etc., etc.} \end{aligned}$$

Therefore the letters with the subscript zero are idempotent and also idemfactorial with respect to all the letters of the same set.

Some of the letters with subscripts other than zero may vanish. If $A_\kappa = 0$, then all the A's with higher subscript vanish, for A_κ contains the factors $(\phi - g_2)^{p_2}(\phi - g_3)^{p_3} \ldots . (\phi - g_s)^{p_s}(\phi - g_1)^\kappa$ besides a homogeneous rational integral function of ϕ and its latent roots, which is non-vacuous as mentioned in Art. 7, and therefore does not affect the vanishing of A_κ; consequently $(\phi - g_2)^{p_2} \ldots . (\phi - g_s)^{p_s}(\phi - g_1)^\kappa = 0$. But this is a factor of all the A's with higher subscripts, they then also vanish.

The A's, B's, etc., are linearly independent. To prove this it is sufficient to show that the A's are linearly independent, for assuming a linear relation between all the letters we have but to multiply it by A_0 to get rid of all the other letters, leaving a linear relation between the A's.

Suppose $A_{p_1 - a_1 - 1}$, $B_{p_2 - a_2 - 1} \ldots . S_{p_s - a_s - 1}$ are the letters with the highest suffices which do not vanish, and suppose the relation

$$a_0 A_0 + a_1 A_1 + a_2 A_2 + \ldots . + a_{p_1 - a_1 - 1} A_{p_1 - a_1 - 1} = 0.$$

Multiply by $(\phi - g_1)^{p_1 - a_1 - 1}$;

$$\therefore \quad a_0 A_0 (\phi - g_1)^{p_1 - a_1 - 1} = 0, \text{ but } A_0(\phi - g_1)^{p_1 - a_1 - 1} \neq 0;$$

$$\therefore \quad a_0 = 0.$$

Similarly all the other coefficients may be shown to be zero and therefore no linear relation exists between the A's or any of the letters.

Now suppose

$$Fg = ag^\kappa + bg^{\kappa - 1} + \ldots . lg + m$$

to be a rational integral function of g, of order κ equal to the order of the lowest equation which ϕ satisfies. Then

$$\begin{aligned}
F\phi &= a\phi^\kappa + b\phi^{\kappa-1} + \ldots . + l\phi + m \\
&= A_0(ag_1^\kappa + bg_1^{\kappa-1} + \ldots . + lg_1 + m) \\
&\quad + A_1(\kappa a g_1^\kappa + \overline{\kappa - 1} b g_1^{\kappa-1} + \ldots . + lg_1) \\
&\quad + A_2(\kappa^2 a g_1^\kappa + \overline{\kappa - 1}^2 b g_1^{\kappa-1} + \ldots . + lg_1) + \text{etc.} \\
&\quad + B_0(ag_2^\kappa + bg_2^{\kappa-1} + \ldots . + lg_2 + m) \\
&\quad + B_1(\kappa a g_2^\kappa + \overline{\kappa - 1} b g_2^{\kappa-1} + \ldots . + lg_2) + \text{etc., etc.} \\
&= A_0 F g_1 + g_1 A_1 F' g_1 + g_1 A_2 (F' g_1 + g_1 F'' g_1) + g_1 A_3 (F' g_1 + 3 g_1 F'' g_1 + g_1^2 F''' g_1) \\
&\quad + g_1 A_4 (F' g_1 + 7 g_1 F'' g_1 + 6 g_1^2 F''' g_1 + g_1^3 F'''' g_1) \\
&\quad + g_1 A_5 (F' g_1 + 15 g_1 F'' g_1 + 25 g_1^2 F''' g_1 + 10 g_1^3 F'''' g_1 + g_1^4 F''''' g_1) \\
&\quad + \text{etc.} \\
&\quad + B_0 F g_2 + g_2 B_1 F' g_2 + g_2 B_2 (F' g_2 + g_2 F'' g_2) + \text{etc.} \\
&\quad \text{etc., etc.}
\end{aligned}$$

$$= A_0 F g_1 + g_1 + g_1 A_1 F' g_1 + \dots + A_i\left(g_1 \frac{d}{dg_1}\right)^i F g_1 + \dots$$
$$+ B_0 F g_2 + g_2 B_1 F' g_2 + \dots + B_j\left(g_2 \frac{d}{dg_2}\right)^j F g_2 + \dots$$
$$+ \text{etc., etc.}$$

Since the A's, B's, etc., are linearly independent, the necessary and sufficient condition that $F\phi = 0$ is

$$F g_1 = 0,\ F' g_1 = 0, \dots F^{(p_1 - \alpha_1 - 1)} g_1 = 0;$$
$$F g_2 = 0,\ F' g_2 = 0, \dots F^{(p_2 - \alpha_2 - 1)} g_2 = 0;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$F g_s = 0,\ F' g_s = 0 \dots F^{(p_s - \alpha_s - 1)} g_s = 0.$$

These results show that the roots of $Fg = 0$ are $g_1, g_2 \dots g_s$, of multiplicities $\overline{p_1 - \alpha_1}$, $\overline{p_2 - \alpha_2}$, ... $\overline{p_s - \alpha_s}$ respectively, and consequently

$$F\phi \equiv (\phi - g_1)^{p_1 - \alpha_1}(\phi - g_2)^{p_2 - \alpha_2} \dots (\phi - g_s)^{p_s - \alpha_s}, \text{ (to a scalar factor),}$$

where ϕ less each of its latent roots occurs as a factor.

II. *Some latent roots zero.*—We now come to the case where some of the latent roots of ϕ are zero.

12. As before, let us consider a few examples and observe the form of the difference equation and its solution.

1). *Matrix of order* 2.

Latent roots g_1, 0.

We obtain the difference equation

$$E(E - g_1)\phi^n = 0, \text{ which gives as solution}$$
$$\phi^n = A_0 g_1^n, \text{ for } n \overset{=}{>} 1;$$
$$\therefore\quad \phi^n = \phi g_1^{n-1}.$$

This is the same value of ϕ^n as would have been obtained by putting $g_2 = 0$ in the expression for ϕ^n, in example 1), of Art. 1.

2). *Matrix of order* 3.

Latent roots g_1, 0, 0.

The difference equation is

$$E^2(E - g_1)\phi^n = 0;$$
$$\therefore\quad \phi^n = A_0 g_1^n, \text{ for } n \overset{=}{>} 2;$$
$$\therefore\quad \phi^n = \phi^2 g_1^{n-2}.$$

This again is the same expression for ϕ^n as we should have obtained by putting $g_2 = g_3 = 0$, in example 2) of Art. 1.

3). *Matrix of order* ω.

Let the latent roots be g_1, and 0 of multiplicity $\overline{\omega - 1}$.

The difference equation is

$$
\begin{aligned}
& E^{\omega-1}(E - g_1)\phi^n = 0\,; \\
\therefore\quad & \phi^n = A_0 g_1^n, \text{ for } n \geqq \overline{\omega - 1}\,; \\
\therefore\quad & \phi^n = \phi^{\omega-1} g_1^{n-\omega+1}.
\end{aligned}
$$

This also is the same expression as we should have obtained by putting $g_2 = g_3 = \ldots\ldots = g_\omega = 0$, in example 3), of Art. 1.

4). *Matrix of order* ω.

Take the most general case, where the latent roots are $g_1,\ g_2 \ldots\ldots g_r,\ 0$, of multiplicities $p_1, p_2, \ldots\ldots p_s$, respectively, and where $p_1 + p_2 + \ldots\ldots + p_s = \omega$.

The difference equation is found to be

$$
(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots\ldots (E - g_r)^{p_r} E^{p_s} \phi^n = 0\,.
$$

For $n \geqq p_s$ the solution of this is

$$
\phi^n = g_1^n \sum_0^{p_1-1}{}^{\lambda} n^\lambda A_\lambda + g_2^n \sum_0^{p_2-1}{}^{\lambda} n^\lambda B_\lambda + \ldots\ldots + g_r^n \sum_0^{p_r-1}{}^{\lambda} n^\lambda R_\lambda .
$$

It is obvious that it makes no difference what values we give to n, in the solution of the difference equation, to obtain expressions for the A's, B's, etc., as long as we take any ω different values which n may have; so that in this case we get for Δ the following:

$$
\Delta = \begin{vmatrix}
g_1^{p_s} & p_s g_1^{p_s} & \ldots\ldots & g_2^{p_s} & \ldots\ldots & g_r^{p_s} & \ldots\ldots & g_r^{p_s} \\
g_1^{p_s+1} & (p_s+1)\, g_1^{p_s+1} & \ldots\ldots & g_2^{p_s+1} & \ldots\ldots & g_r^{p_s+1} & \ldots\ldots & (p_s+1)^{p_r-1} g_r^{p_s+1} \\
\vdots & \vdots & \ldots\ldots & \vdots & \ldots\ldots & \vdots & \ldots\ldots & \vdots \\
\vdots & \vdots & \ldots\ldots & \vdots & \ldots\ldots & \vdots & \ldots\ldots & \vdots \\
g_1^{\omega-1} & (\omega-1)\, g_1^{\omega-1} & \ldots\ldots & g_2^{\omega-1} & \ldots\ldots & g_r^{\omega-1} & \ldots\ldots & (\omega-1)^{p_r-1} g_r^{\omega-1}
\end{vmatrix}.
$$

If we examine the equations from which the A's, B's, etc., are determined, we shall easily see that the A's, B's, etc., of this case are what the A's, B's, etc., of example 3), Art. 7, become, when g_s is put equal to zero; and consequently the expression for ϕ^n found here is what that found in example 3), Art. 7, reduces to when $g_s = 0$.

For $n < p_s$, the solution of the difference equation is

$$\phi^n = g_1^n \sum_{0}^{p_1-1}{}_{\lambda}\, n^\lambda A_\lambda + \ldots + g_r^n \sum_{0}^{p_r-1}{}_{\lambda}\, n^\lambda R_\lambda + N_n,$$

where N_n is some expression resulting from the solution of $E^{p_s}\phi^n = 0$, and may be determined in the same way as the expressions for the A's, B's, etc. When $n \geqq p_s$, $N_n = 0$.

If the expression for N_n be determined, as just mentioned, it will be found to be what the term $g_s^n \sum_{0}^{p_s-1}{}_{\lambda}\, n^\lambda S_\lambda$ reduces to when $g_s = 0$. This term does not vanish when g_s becomes zero, as might appear.

We have $\quad g_s^n S_0 = 0$, when $n \geqq 1$ and $g_s = 0$;
and $\quad g_s^n S_\lambda = 0$, " $n \geqq p_s$ " $g_s = 0$;
as may easily be seen from their factors given in Art. 7.

That N_n is what this term reduces to when $g_s = 0$, is also apparent from the fact that the solution of

$$(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots (E - g_r)^{p_r} E^{p_s}\phi^n = 0$$

is the same as what would result from putting $g_s = 0$ in the solution of

$$(E - g_1)^{p_1}(E - g_2)^{p_2} \ldots (E - g_r)^{p_r}(E - g_s)^{p_s}\phi^n = 0.$$

We arrive at the conclusion, therefore, that the formula for ϕ^n when none of its latent roots are zero, still applies when some of them become zero.

13. *Expression for unity.*—In the cases where none of the latent roots were zero we could put $n = 0$ in the solution of the difference equation and obtain an expression for unity, but when ϕ is vacuous we cannot do so. In the general case when none of the latent roots are zero we have

$$1 = A_0 + B_0 + C_0 + \ldots + S_0,$$

which is an identity; that is, if developed according to powers of ϕ, the coefficient of each power of ϕ (which is a function of the latent roots) would be identically zero. This property of the coefficients still subsists however small any of the latent roots become, and consequently is still true when one of them becomes zero. Suppose $g_s = 0$, and denote by $A_0', B_0' \ldots . S_0'$ what $A_0, B_0 \ldots . S_0$ become. We have then

$$1 = A_0' + B_0' + \ldots \ldots + S_0',$$

which is an expression for unity when some of the latent roots are zero.

14. *Powers of ϕ.*—The difference equation gives us an expression for ϕ^n; but ϕ, being vacuous, has no reciprocal and therefore n can have only positive values. This formula for ϕ^n gives a means of expressing the ω^{th} and higher powers of ϕ as rational integral functions of the $(\omega - 1)^{\text{st}}$ and lower powers.

15. *Rational integral function of ϕ.*—Having an expression for any positive integral power of ϕ, we can write any rational integral function of an order not less than ω as follows:

$$\sum_0^n {}^{\mu} a_\mu \phi^\mu = \sum_0^n {}^{\mu} a_\mu \Big[g_1^\mu \sum_0^{p_1 - 1} {}^{\lambda} \mu^\lambda A_\lambda + \ldots . + g_r^\mu \sum_0^{p_r - 1} {}^{\lambda} \mu^\lambda R_\lambda + N_\mu \Big].$$

16. *Identical Equation.*—We have

$$\phi^\omega = (A_0 + \omega A_1 + \text{etc.}) g_1^\omega + (B_0 + \omega B_1 + \text{etc.}) g_2^\omega + \ldots . + (R_0 + \omega R_1 + \text{etc.}) g_r^\omega;$$

and it has been observed that the A's, B's, etc., here are what those of Art. 11 reduce to when $g_s = 0$. This expression for ϕ^ω is therefore what the expression of that article reduces to when $g_s = 0$, and therefore it may be written

$$(\phi - g_1)^{p_1} (\phi - g_2)^{p_2} \ldots . (\phi - g_r)^{p_r} \phi^{p_s} = 0,$$

which is the identical equation unless ϕ satisfies an equation of lower order.

Having this equation, the factors of the various letters show that (1) the letters with subscripts other than zero are nilpotent and (2) the letters of any one set are nilfactorial with respect to the letters of any other set. From the expression for unity, as given in Art. 13, it may be shown in the same manner as in Art. 11, that the letters with subscript zero are idempotent and also idemfactorial with respect to all the other letters of the same set.

Each one of the sum of S's of which N_μ is composed contains all the factors of the identical equation, and consequently N_μ is nilpotent.

It may easily be shown that all the letters, including N_μ, are linearly independent; and we have therefore sufficient data for showing in precisely the same manner as it was shown in Art. 11, that ϕ less each of its latent roots must appear as a factor in the identical equation; and also that ϕ satisfies no other equation of the same order. The identical equation may then be written

$$(\phi-g_1)^{p_1-a_1}(\phi-g_2)^{p_2-a_2}\ldots\ldots(\phi-g_r)^{p_r-a_r}\phi^{p_s-a_s}=0.$$

§2.—*Properties of the A's, B's, etc.*

17. We have already proven that:

(1). The letters with the subscript zero are idempotent and idemfactorial with respect to all the other letters of the same set;

(2). The letters of any one set are nilfactorial with respect to the letters of any other set; and

(3). The letters with subscripts other than zero are nilpotent;

and we shall now establish a relation between the letters of any set.

18. *Relation between the A's.*—All the letters of any set, with subscripts greater than unity, can be expressed as powers of the letter with subscript unity according to the following law:

$$L_1^\lambda=\lambda!\ L_\lambda.$$

This relation may be established in two different ways as follows:

First Method.

We may write

$$\begin{aligned}A_0\phi&=A_0[(\phi-g_1)+g_1]\\&=A_0[A_0(\phi-g_1)+g_1]\\&=A_0g_1\Big[\frac{A_0(\phi-g_1)}{g_1}+1\Big]\\&=A_0g_1[N+1],\end{aligned}$$

where N is put for $\dfrac{A_0(\phi-g_1)}{g_1}$.

Let us denote by $\log(N+1)$ the series

$$N-\frac{N^2}{2}+\frac{N^3}{3}-\frac{N^4}{4}+\ldots\ldots+(-1)^{n-1}\frac{N^n}{n}+\text{etc.},$$

which is finite, since the p_1^{st} and all higher powers of N vanish.

If we develop $e^{\log(N+1)} = e^{N-\frac{N^2}{2}+\frac{N^3}{3}-\text{etc.}}$ according to ascending powers of N, we shall find that the coefficient of N is unity, that the coefficients of the 2nd and of all higher powers as far as we choose to go vanish, and we know that all terms containing the p_1^{st} and higher powers vanish, because N^{p_1} contains the identical equation as a factor.

We may therefore write

$$\begin{aligned} 1 + N &= e^{\log(1+N)}; \\ \therefore \quad A_0\phi &= A_0 g_1 e^{\log(1+N)} \\ &= A_0 g_1 e^{A}, \end{aligned}$$

where A is put for $\log(1+N)$.

$$\begin{aligned} A_0\phi^n &= A_0 g_1^n e^{nA}; \\ \therefore \quad A_0\phi^n &= A_0 g_1^n \left(1 + nA + \frac{n^2A^2}{2} + \frac{n^3A^3}{3!} + \text{etc.}\right) \\ &= g_1^n \left(A_0 + nA + \frac{n^2A^2}{2} + \frac{n^3A^3}{3!} + \text{etc.}\right), \end{aligned}$$

A_0 being idemfactorial with respect to the A^λ's.

The expression already found for ϕ^n is

$$\begin{aligned} \phi^n &= (A_0 + nA_1 + n^2A_2 + \text{etc.})g_1^n + (B_0 + nB_1 + \text{etc.})g_2^n + \text{etc.}; \\ \therefore \quad A_0\phi^n &= (A_0 + nA_1 + n^2A_2 + \text{etc.})g_1^n. \end{aligned}$$

These two expressions must be identical;

$$\therefore \quad A_0 + nA_1 + \text{etc.} + n^{p_1-1}A_{p_1-1} \equiv A_0 + nA + \frac{n^2A^2}{2} + \ldots + \frac{(nA)^{p_1-1}}{(p_1-1)!}.$$

It is obvious that $A^{p_1} = 0$, since $N^{p_1} = 0$;

$$\begin{aligned} \therefore \quad (A - A_1) + n\left(\frac{A^2}{2!} - A_2\right) + n^2\left(\frac{A^3}{n!} - A_3\right) + n^3\left(\frac{A^4}{4!} - A_4\right) + \text{etc.} \\ + n^{p_1-2}\left(\frac{A^{p_1-1}}{(p_1-1)!} - A_{p_1-1}\right) \equiv 0. \end{aligned}$$

In any case this is true for all positive integral values of n, and therefore the coefficients of the various powers of n must be identically zero. We have then

$$A \equiv A_1 \text{ and generally } A^\lambda \equiv \lambda\,!\,A_\lambda.$$

Having all the A's expressed in terms of A_1 we can find the relations between them; and in an exactly similar manner the same relation may be found to exist between B's, C's, etc.

This proof of the relation is due to Dr. Henry Taber. I have found a proof of it based on the fact that

$$(A_0 + A_1 + \text{etc.})^\lambda \equiv A_0 + \lambda A_1 + \lambda^2 A_2 + \text{etc.},$$

which is somewhat long and complicated but not without some interest. It is as follows:

Second method.

For convenience suppose we have a matrix ϕ of order ω whose latent roots are $g_1, g_2 \ldots . g_s$ occurring $p_1 + 1, p_2 + 1, \ldots . p_s + 1$ times respectively. Write

$\overset{2}{a}_2$ for $2^2 - 2$,

$\overset{3}{a}_3$ " $3^3 - 3 - 3(2^3 - 2)$,

$\overset{4}{a}_4$ " $4^4 - 4 - \frac{4.3}{2}\{3^4 - 3 - 3(2^4 - 2)\} - 4\{2^4 - 2\}$,

$\vdots$

$\overset{p_1}{a}_{p_1}$ " $p_1^{p_1} - p_1 - \frac{p_1!}{(p_1 - 3)!\,3!}\overset{p_1}{a}_2 - \text{etc.} \ldots . - \frac{p_1!}{(p_1 - k)!\,k!}\overset{p_1}{a}_\kappa - \ldots . - \frac{p_1!}{(p_1 - 1)!}\overset{p_1}{a}_{p_1 - 1}$,

where $\overset{\lambda}{a}_2 = 2^\lambda - 2$,

$\overset{\lambda}{a}_3 = 3^\lambda - 3 - 3(2^\lambda - 2)$, etc., etc.;

also write

$$T \text{ for } A_1 + A_2 + \ldots . + Ap_1.$$

It is readily seen that

$\overset{2}{a}_2 = 2^2 - 2$,

$\overset{3}{a}_3 = (3^3 - 3) - 3(2^3 - 2)$,

$\overset{4}{a}_4 = (4^4 - 4) - 4(3^4 - 3) + \frac{4.3}{2}(2^4 - 2)$,

$\overset{5}{a}_5 = (5^5 - 5) - 5(4^5 - 4) + \frac{5.4}{2}(3^5 - 3) - \frac{5.4.3}{2.3}(2^5 - 2)$,

$\vdots$

$$\overset{p}{a}_p = (p^p - p) - p(\overline{p - 1}^p - \overline{p - 1}) + \ldots . + (-1)^\kappa \frac{p!}{(p - \kappa)!\,\kappa!}(\overline{p - \kappa}^p - \overline{p - \kappa}) \ldots .$$
$$\ldots . + (-1)^p \frac{p(p - 1)}{2}(2^p - 2),$$

where the subscript of p has been dropped for convenience, since no confusion can thereby arise.

Separating the value of $\overset{p}{a}_p$ into two parts we get

$$\begin{aligned}\overset{p}{a}_p &= p^p - p(p-1)^p + \ldots\ldots + (-1)^\varkappa \frac{p!}{(p-\varkappa)!\,k!}(p-\varkappa)^p + \ldots\ldots + (-1)^p \frac{p(p-1)}{2}2^p \\ &\quad - p + p(p-1) + \ldots\ldots - (-1)^\varkappa \frac{p!}{(p-\varkappa-1)!\,k!} - \ldots\ldots - (-1)^p p(p-1) \\ &= p! + p(1-1)^{p-1} \\ &= p!\,.\end{aligned}$$

We know that

$$\overset{\varkappa}{a}_n = 0 \text{ for } \varkappa < n,$$

$$\overset{n}{a}_n = n!\,,$$

$$\overset{n+1}{a}_n = \tfrac{1}{2}n(n+1)!\,,$$

$$\overset{n+2}{a}_n = \left\{\frac{n(n-1)}{2^3} + \frac{n}{3!}\right\}(n+2)!\,,$$

$$\vdots$$

$$\begin{aligned}\overset{n+\lambda}{a}_n = \Big[\frac{n}{(\lambda+1)!} &+ \frac{n(n-1)}{2}\left\{\frac{1}{(\lambda+2)!} + \frac{2}{2!\,\lambda!} + \text{etc.}\right\} \\ &+ \frac{n(n-1)(n-2)}{3!}\left\{\frac{1}{(\lambda+3)!} + \frac{3}{(\lambda-1)!\,(2!)^2} + \text{etc.}\right\} \\ + \ldots\ldots + \frac{n!}{(\varkappa+1)!\,(n-\varkappa-1)}&\left\{\frac{1}{(\lambda+k+1)!} + \frac{\varkappa+1}{(2!)^\varkappa\,(\lambda-\varkappa+1)!} + \text{etc.}\right\} \\ &+ \text{etc.}\Big]\,(n+\lambda)!\end{aligned}$$

$=$ coefficient of $x^{n+\lambda}$ in the expansion of $(e^x-1)^n = \left(\frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)^n$, multiplied by $(n+\lambda)!$.

Having obtained the foregoing auxiliary theorems we may proceed to the more direct consideration of the relation between the letters.

Taking the various powers of T up to the p^{th} we get

1). $T = A_1 + A_2 + \ldots\ldots + A_p$

$\therefore \quad T^2 + 2T = 2A_1 + 2^2A_2 + \ldots\ldots + 2^pA_p,$

$\therefore$

2). $T^2 = 2!\,A_2 + \overset{3}{a}_2A_3 + \ldots\ldots + \overset{p}{a}_2A_p \qquad = \Sigma A_1^2 + 2\Sigma A_1A_2,$

$$3).\quad T^3 = 3\,!\,A_3 + \overset{4}{a}_3 A_4 + \ldots\ldots + \overset{p}{a}_3 A_p \qquad = \Sigma A_1^3 + 3\Sigma A_1^2 A_2 + 6\Sigma A_1 A_2 A_3,$$

$$\vdots$$

$$n).\quad T^n = n\,!\,A_n + \overset{n+1}{a}_n A_{n+1} + \ldots\ldots + \overset{p}{a}_n A_p \qquad = \Sigma A_1^n + n\Sigma A_1^{n-1} A_2 + \text{etc.},$$

$$\vdots$$

$$p).\quad T^p = p\,!\,A_p \qquad = A_1^p.$$

From these equations we easily get, on multiplying by proper powers of the various A's,

$$A_1^p = (p-\kappa)A_1^\kappa A_{p-\kappa} \qquad = \kappa\,!\,(p-\kappa)\,!\,A_{p-\kappa}A_\kappa,$$

$$A_1^{p-\lambda\mu}A_\lambda^\mu = (p-\mu\lambda)\,!\,A_{p-\lambda\mu}A_\lambda^\mu,$$

etc.

$$A_1^p = (\kappa_1\,!)^{a_1}(\kappa_2\,!)^{a_2}\ldots\ldots(\kappa_m\,!)^{a_m}(p-\kappa_1^{a_1}-\kappa_2^{a_2}-\ldots\ldots-\kappa_m^{a_m})\,!\,A_{\kappa_1}^{a_1}\ldots\ldots A_{p-\kappa_1^{a_1}\ldots\ldots\kappa_m^{a_m}}.$$

In this way we get all relations between expressions of weight p.

From equation $p-1$) we get

$$A_1^{p-1} + (p-1)A_1^{p-2}A_2 = (p-1)\,!\,A_{p-1} + \tfrac{1}{2}(p-1)p\,!\,A_p.$$

$$\text{But } A_1^{p-2}A_2 = \tfrac{1}{2}p\,!\,A_p;$$

$$\therefore\quad A_1^{p-1} = (p-1)\,!\,A_{p-1}.$$

Similarly, as in case of weight p, all relations between expressions of weight $(p-1)$ can be found.

$$A_1^{p-\lambda}A_{\lambda-1} = (p-\lambda)\,!\,A_{p-\lambda}A_{\lambda-1},$$

etc. etc.

$$A_1^{p-1} = (\kappa_1\,!)^{a_1}(\kappa_2\,!)^{a_2}\ldots\ldots(\kappa_m\,!)^{a_m}(p-\kappa_1^{a_1}-\kappa_2^{a_2}\ldots\ldots\kappa_m^{a_m}-1)\,!\,A_{\kappa_1}^{a_1}\ldots\ldots A_{p-\kappa_1^{a_1}\ldots\ldots\kappa_m^{a_m}-1}.$$

From equation $p-2$) we get

$$A_1^{p-2} + (p-2)A^{p-3}(A_2+A_3) + \frac{(p-2)(p-3)}{2}A_1^{p-4}A_2^2$$

$$= (p-2)\,!\,A_{p-2} + \tfrac{1}{2}(p-2)(p-1)\,!\,A_{p-1} + \left\{\frac{p-2}{3\,!} + \frac{(p-2)(p-3)}{2^3}\right\}p\,!\,A_p;$$

$$\therefore\quad A_1^{p-2} = (p-2)\,!\,A_{p-2}.$$

Again, we could proceed as before and obtain all relations between expressions of weight $(p-2)$.

Suppose we have performed all the operations and found all the relations between expressions of the same weight to $A_1^{n+1} = (n+1)\,!\,A_{n+1}$.

Then let us consider equation n).—

$$A_1^n + nA_1^{n-1}(A_2 + A_3 + \text{etc.}) + \frac{n(n-1)}{2}A_1^{n-2}(A_2 + A_3 + \text{etc.})^2 + \text{etc.}$$
$$= n!\,A_n + \tfrac{1}{2}n(n+1)!\,A_{n+1} + \ldots. + \overset{n+\lambda}{a_n}A_{n+\lambda} + \ldots. + \overset{p}{a_n}A_p.$$

I wish now to show that the terms of weight $(n+\lambda)$ on the one side of the equation cancel those of the same weight on the other side, where λ may have any integral value from 1 to $p-n$.

Collecting the terms of weight $n+\lambda$ on the left hand side of the equation we get

$$nA_1^{n-1}A_{\lambda+1} + \frac{n(n-1)}{2}A_1^{n-2}\Big(\overset{*}{A}{}^{2}_{\frac{\lambda+2}{2}} + 2A_2A_\lambda + 2A_3A_{\lambda-1} + \text{etc.}\Big)$$
$$+ \frac{n(n-1)(n-2)}{3!}A_1^{n-3}\Big(\overset{*}{A}{}^{3}_{\frac{\lambda+3}{3}} + 3\Sigma A_2^2A_{\lambda-1} + \text{etc.}\Big)$$
$$\ldots. + \frac{n!}{(\kappa+1)!\,(n-\kappa+1)!}A_1^{n-\kappa-1}\Big(\overset{*}{A}{}^{\kappa+1}_{\frac{\lambda+\kappa+1}{\kappa+1}} + (\kappa+1)\Sigma A_2^\kappa A_{\lambda-\kappa+1} + \text{etc.}\Big) + \text{etc.}$$

Replacing each of the terms in this by the proper function of $A_1^{n+\lambda}$ we get

$$\Big[\frac{n}{(\lambda+1)!} + \frac{n(n-1)}{2}\Big\{\frac{1}{(\lambda+2)!} + \frac{2}{2!\,\lambda!} + \frac{2}{3!\,(\lambda-1)!} + \text{etc.}\Big\}$$
$$+ \frac{n(n-1)(n-2)}{3!}\Big\{\frac{1}{(\lambda+3)!} + \frac{3}{(2!)^3(\lambda-1)!} + \ldots. + \frac{6}{2!\,\kappa!\,(\lambda-\kappa)!} + \text{etc.}\Big\}$$
$$+ \ldots. + \frac{n!}{(\kappa+1)!\,(n-\kappa+1)!}\Big\{\frac{1}{(\lambda+\kappa+1)!} + \frac{\kappa+1}{(2!)^\kappa(\lambda-\kappa+1)!} + \text{etc.}\Big\}$$
$$+ \text{etc.} \ldots. \Big]A_1^{n+\lambda}.$$

This is at once seen to be $\dfrac{\overset{n+\lambda}{a_n}}{(n+\lambda)!}A_1^{n+\lambda}$.

But $A_1^{n+\lambda} = (n+\lambda)!\,A_{n+\lambda}$.

Therefore the term of weight $n+\lambda$ on the left is equal to the term of weight $n+\lambda$ on the right which is $\overset{n+\lambda}{a_n}A_{n+\lambda}$;

$$\therefore\quad A_1^n = n!\,A_n,$$

where n may have any value from 2 to p, and therefore our theorem is established.

In a similar manner the same relation may be shown to exist between the letters B, C, etc.

---

* These terms will not appear unless $\frac{\lambda+h}{h}$ is an integer.

19. *Relation between the N_μ.*—We have already observed that N_μ was nilpotent and nilfactorial with respect to all the other letters, and, knowing the foregoing properties of the A's, B's, etc., we may now find the relation between the N_μ of Art. 12.

$$\begin{aligned}\phi &\equiv (A_0 + A_1 + \text{etc.})g_1 + (B_0 + B_1 + \text{etc.})g_2 + \dots + (R_0 + R_1 + \text{etc.})g_r + N_1,\\ \phi^2 &\equiv (A_0 + 2A_1 + \text{etc.})g_1^2 + (B_0 + 2B_1 + \text{etc.})g_2^2 + \dots + (R_0 + 2R_1 + \text{etc.})g_r^2 + N_2\\ &\equiv \{(A_0 + A_1 + \text{etc.})g_1 + \dots + (R_0 + R_1 + \dots)g_r + N_1\}^2\\ &\equiv (A_0 + 2A_1 + \text{etc.})g_1^2 + (B_0 + 2B_1 + \text{etc.})g_2^2 + \dots + (R_0 + 2R_1 + \text{etc.})g_r^2 + N_1^2;\end{aligned}$$

$\therefore \quad N_2 \equiv N_1^2$, and generally it will be found that $N_\mu \equiv N_1^\mu$.

I shall hereafter omit the subscript of N_1.

§3.—*Law of Latency.*

20. I shall consider the two cases, first where none of the latent roots are zero, and second where some latent roots are zero.

First case.—Suppose $A_{p_1-a_1}$, $B_{p_2-a_2} \dots S_{p_s-a_s}$ are the letters with greatest subscripts which do not vanish. The rational function of Art. 9 may be written as follows:

$$\sum_{0}^{n}{}^{\mu} a_\mu \phi^\mu = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu A_0 + g_1^\mu \sum_{1}^{p_1-a_1}{}^{\lambda} \mu^\lambda A_\lambda + \dots + g_s^\mu S_0 + g_s^\mu \sum_{1}^{p_s-a_s}{}^{\lambda} \mu^\lambda S_\lambda \Big];$$

then, writing $\sum_{0}^{n}{}^{\mu} a_\mu \phi^\mu = f\phi$, we have

$$\begin{aligned}(f\phi - fg_1) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu \sum_{1}^{p_1-a_1}{}^{\lambda} \mu^\lambda A_\lambda + (g_2^\mu - g_1^\mu) B_0 + g_2^\mu \sum_{1}^{p_2-a_2}{}^{\lambda} \mu^\lambda B_\lambda + \text{etc.}&\\ + (g_s^\mu - g_1^\mu) S_0 + g_s^\mu \sum_{1}^{p_s-a_s}{}^{\lambda} \mu^\lambda S_\lambda \Big]&,\end{aligned}$$

$$(f\phi - fg_2) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_2^\mu) A_0 + g_1^\mu \sum_{0}^{p_1-a_1}{}^{\lambda} \mu^\lambda A_\lambda + \dots + (g_s^\mu - g_2^\mu) S_0 + g_s^\mu \sum_{1}^{p_s-a_s}{}^{\lambda} \mu^\lambda S_\lambda \Big],$$

$$\vdots$$

$$(f\phi - fg_s) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_s^\mu) A_0 + \dots + g_s^\mu \sum_{1}^{p_s-a_s}{}^{\lambda} \mu^\lambda S_\lambda \Big].$$

In the first equation A_0, in the second B_0, in the third C_0, etc. and in the last S_0 do not appear.

Again

$$\begin{array}{lllll}
(f\phi - fg_1)^{p_1-\alpha_1} & \text{obviously} & \text{contains} & \text{none of the} & A\text{'s}, \\
(f\phi - fg_2)^{p_2-\alpha_2} & \text{“} & \text{“} & \text{“} \quad \text{“} & B\text{'s}, \\
\vdots & & & & \\
(f\phi - fg_s)^{p_s-\alpha_s} & \text{“} & \text{“} & \text{“} \quad \text{“} & S\text{'s}, \\
(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2} & \text{“} & \text{“} & \multicolumn{2}{l}{\text{neither } A\text{'s nor } B\text{'s},}
\end{array}$$

but contains all the other letters,

$(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2} \ldots . (f\phi - fg_r)^{p_r-\alpha_r}$ contains none of the letters but S's, and

$$(f\phi - fg_1)^{p_1-\alpha_1}(f\phi - fg_2)^{p_2-\alpha_2} \ldots . (f\phi - fg_s)^{p_s-\alpha_s} = 0.$$

The above expressions of the type $(f\phi - fg)$ and their products and powers are evidently the expressions of lowest orders in $f\phi$ that have the characters specified viz. as to the absence of the A's, B's, etc.

The latent roots of $f\phi$ are $fg_1, fg_2 \ldots . fg_s$ of multiplicities at least $p_1 - \alpha_1$, $p_2 - \alpha_2, \ldots, p_s - \alpha_s$ respectively.

If $\alpha_1 = \alpha_2 = \ldots . = \alpha_s = 0$, then the latent roots of $f\phi$ are $fg_1, fg_2, \ldots . fg_s$ of multiplicities $p_1, p_2 \ldots . p_s$ respectively, since $\sum_{1}^{s}{}^{\lambda} p_\lambda = \omega$, and $f\phi$ has ω latent roots.

Second Case.—Suppose, as in the previous case, that $A_{p_1-\alpha_1}, B_{p_2-\alpha_2} \ldots . R_{p_r-\alpha_r}$ are the letters with greatest subscripts which do not vanish and that $N^{p_s-\alpha_s}$ is the greatest power of N that does not vanish. The rational integral function of Art. 15 may be written as follows:

$$F\phi = \sum_{0}^{n}{}^{\mu} a_\mu \phi^\mu = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu A_0 + g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + \ldots . + g_r^\mu \sum_{1}^{p_r-\alpha_r}{}^{\lambda} \mu^\lambda R_\lambda + N^\mu \Big].$$

Making use of the expression for unity and proceeding as before we have

$$(F\phi - Fg_1) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + (g_2^\mu - g_1^\mu) B_0 + \text{etc.} \ldots . - S_0 g_1^\mu + N^\mu \Big],$$

$$(F\phi - Fg_2) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_2^\mu) A_0 + g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + g_2^\mu \sum_{1}^{p_2-\alpha_2}{}^{\lambda} \mu^\lambda B_\lambda + \ldots . - S_0 g_2^\mu + N^\mu \Big],$$

$$\vdots$$

$$(F\phi - Fg_r) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[(g_1^\mu - g_r^\mu) A_0 + \ldots . + g_r^\mu \sum_{1}^{p_r-\alpha_r}{}^{\lambda} \mu^\lambda R_\lambda - S_0 g_r^\mu + N^\mu \Big],$$

$$(F\phi - F_0) = \sum_{0}^{n}{}^{\mu} a_\mu \Big[g_1^\mu A_0 + g_1^\mu \sum_{1}^{p_1-\alpha_1}{}^{\lambda} \mu^\lambda A_\lambda + \ldots . + g_r^\mu \sum_{1}^{p_r-\alpha_r}{}^{\lambda} \mu^\lambda R_\lambda + N^\mu \Big] \cdot$$

A_0 disappears from the first equation,
B_0 " " second "
⋮
S_0 " " last "

$(F\phi - Fg_1)^{p_1-\alpha_1}$ obviously contains none of the A's,
$(F\phi - Fg_2)^{p_2-\alpha_2}$ " " " " B's,
⋮
$(F\phi - F_0)^{p_s-\alpha_s}$ " does not contain N;

and as before,

$$(F\phi - Fg_1)^{p_1-\alpha_1}(F\phi - Fg_2)^{p_2-\alpha_2}\ldots\ldots(F\phi - Fg_r)^{r-\alpha_r}(F\phi - F_0)^{p_s-\alpha_s} = 0.$$

The above expressions of the type $(F\phi - Fg)$ and their powers and products are evidently the expressions of lowest orders in $F\phi$ that have the characters specified viz., as to the absence of the A's, B's, etc.

The latent roots of $F\phi$ are Fg_1, $Fg_2 \ldots\ldots Fg_r$, F_0, of multiplicities at least $p_1 - \alpha_1$, $p_2 - \alpha_2, \ldots\ldots p_s - \alpha_s$ respectively.

If $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_s = 0$ then the latent roots of $F\phi$ are $Fg_1, Fg_2, \ldots\ldots Fg_r$, F_0, of multiplicities $p_1, p_2, p_3, \ldots\ldots p_s$ respectively, since $\sum_1^s p_\lambda = \omega$, and $F\phi$ has ω latent roots.

§4.—*Nullity of the factors of the identical equation.*

21. Let the identical equation be

$$(\phi - g_1)^{p_1-\alpha_1}(\phi - g_2)^{p_2-\alpha_2}\ldots\ldots(\phi - g_r)^{r-\alpha_r}\phi^{p_s-\alpha_s} = 0.$$

Denote all the factors of this equation except the first by ψ_1.

Then

$$(\phi - g_1)^{p_1-\alpha_1}\psi_1 = 0,$$

$N_y[\phi - g_1)^{p_1-\alpha_1}\psi_1] = \omega$, where $N_y[\phi]$ denotes "the nullity of ϕ".

But $N_y[\psi_1] \stackrel{=}{<} \omega - p_1$, since vacuity of ψ_1 is $\omega - p_1$,

and $N_y[(\phi - g_1)^{p_1-\alpha_1}] \stackrel{=}{<} p_1$;

$\therefore\ N_y[\psi_1] = \omega - p_1$,

and $N_y[(\phi - g_1)^{p_1-\alpha_1}] = p_1$.

Similarly, $N_y[(\phi - g_\lambda)^{p_\lambda-\alpha_\lambda}] = p_\lambda$.

Let

$$\psi_1 = (\phi - g_2)^{p_2 - a_2}\psi_2.$$

We know that

$$N_y[\psi_1] = \omega - p_1,$$

or $$N_y[(\phi - g_2)^{p_2 - a_2}\psi_2] = \omega - p_1.$$

But $$N_y[(\phi - g_2)^{p_2 - a_2}] = p_2 \text{ and}$$

$$N_y[\psi_2] \overset{=}{<} \omega - p_1 - p_2, \text{ since the vacuity of } \psi_2 \text{ is } \omega - p_1 - p_2;$$

$$\therefore \quad N_y[\psi_2] = \omega - p_1 - p_2.$$

Again

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}] \overset{=}{<} p_1 + p_2,$$

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}\psi_2] = \omega;$$

$$\therefore \quad N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2}] = p_1 + p_2:$$

and generally

$$N_y[(\phi - g_1)^{p_1 - a_1}(\phi - g_2)^{p_2 - a_2} \ldots . (\phi - g_\lambda)^{p_\lambda - a_\lambda}] = p_1 + p_2 + \ldots . + p_\lambda.$$

§5.—*Roots of a Matrix.*

Under this head I shall distinguish two cases,

I. When the latent roots are all different from zero and

II. When some latent roots are zero.

I.—*Latent roots* $\neq 0$.

22. Knowing the expression for the A's, B's, etc., and the relations between them, we may write the expression for ϕ^n as follows:

$$\phi^n = A_0 g_1^n e^{nA_1} + B_0 g_2^n e^{nB_1} + \ldots . S_0 g_s^n e^{nS_1}.$$

Writing in this formula $n = \frac{1}{m}$ we get

$$\phi^{\frac{1}{m}} = A_0 g_1^{\frac{1}{m}} e^{\frac{A_1}{m}} + B_0 g_2^{\frac{1}{m}} e^{\frac{B_1}{m}} + \ldots . + S_0 g_s^{\frac{1}{m}} e^{\frac{S_1}{m}}.$$

Taking the m^{th} power of both sides we get

$$\phi = A_0 g_1 e^{A_1} + B_0 g_2 e^{B_1} + \ldots . + S_0 g_s e^{S_1};$$

$\therefore$ ϕ has an m^{th} root.

23. In the formula for $\phi^{\frac{1}{m}}$ we have s different m-valued functions, viz. $g_1^{\frac{1}{m}}, g_2^{\frac{1}{m}}, \ldots . g_s^{\frac{1}{m}}$, and consequently taking all possible combinations of these values we get m^s different m^{th} roots of ϕ.

24. It may be interesting to consider a few examples to show something of the character of the various functions F entering as factors in the A's, B's, etc.

1). *Matrix of order* 5.

Latent roots g_1 of multiplicity five.

$$\phi = A_0 g_1^n e^{nA_1} = (A_0 + nA_1 + n^2A_2 + n^3A_3 + n^4A_4)g_1^n,$$

$$\Delta = 4!\ 3!\ 2!\ g_1^{10},$$

$$A_0 = 1,$$

$$A_1 = -\frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{3g_1^3} - \frac{(\phi - g_1)^2}{2g_1^2} + \frac{(\phi - g_1)}{g_1},$$

$$A_2 = \frac{11}{24}\frac{(\phi - g_1)^4}{g_1^4} - \frac{(\phi - g_1)^3}{2g_1^3} + \frac{(\phi - g_1)^2}{2g_1^2},$$

$$A_3 = -\frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{6g_1^3},$$

$$A_4 = \frac{(\phi - g_1)^4}{24g_1^4}.$$

2). *Matrix of order* 6.

Latent roots all equal.

$$\phi^n = A_0 g_1^n e^{nA_1},$$

$$\Delta = 5!\ 4!\ 3!\ 2!\ g_1^{15},$$

$$A_0 = 1,$$

$$A_1 = \frac{(\phi - g_1)^5}{5g_1^5} - \frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{3g_1^3} - \frac{(\phi - g_1)^2}{2g_1^2} + \frac{(\phi - g_1)}{g_1},$$

$$A_2 = -\frac{5(\phi - g_1)^5}{12g_1^5} + \frac{11(\phi - g_1)^4}{24g_1^4} - \frac{(\phi - g_1)^3}{2g_1^3} + \frac{(\phi - g_1)^2}{2g_1^2},$$

$$A_3 = \frac{7(\phi - g_1)^5}{24g_1^5} - \frac{(\phi - g_1)^4}{4g_1^4} + \frac{(\phi - g_1)^3}{6g_1^3},$$

$$A_4 = -\frac{(\phi - g_1)^5}{12g_1^5} + \frac{(\phi - g_1)^4}{24g_1^4},$$

$$A_5 = \frac{(\phi - g_1)^5}{5!\ g_1^5}.$$

3). *Matrix of order* ω.

Latent roots all equal.

$$\phi = A_0 g_1^n e^{nA_1},$$

$$\Delta = (\omega - 1)!\ (\omega - 2)!\ \ldots\ 2!\ g_1^{\frac{\omega(\omega-1)}{2}},$$

$$A_0 = 1,$$

$$A_1 = (-1)^{\omega} \frac{(\phi - g_1)^{\omega-1}}{(\omega-1)g_1^{\omega-1}} + (-1)^{\omega-1} \frac{(\phi - g_1)^{\omega-2}}{(\omega-2)g_1^{\omega-2}} + \ldots\ldots + \frac{(\omega - g_1)}{g_1},$$

$$\vdots$$

$$A_{\omega-1} = \frac{(\phi - g_1)^{\omega-1}}{(\omega-1)!\ g_1^{\omega-1}}.$$

4). *Matrix of order* 4.

Latent roots $g_1 = g_3 = g_4,\ g_2$.

$$\phi^n = (A_0 + nA_1 + n^2A_2)g_1^n + B_0 g_2^n,$$

$$\Delta = 2!\ g_1^3(g_1 - g_2)^3,$$

$$A_0 = \frac{(\phi - g_2)\{(\phi - g_1)^2 + (\phi - g_1)(g_2 - g_1) + (g_2 - g_1)^2\}}{(g_1 - g_2)^3},$$

$$B_0 = \frac{(\phi - g_1)^3}{(g_2 - g_1)^3}.$$

5). *Matrix of order* ω.

Latent roots $g_1 = g_3 = g_4 = \ldots\ldots = g_{p+1},\ g_2 = g_{p+2} = \ldots\ldots = g_{p+q}$, where $p + q = \omega$.

$$\phi^n = A_0 g_1^n e^{nA_1} + B_0 g_2^n e^{nB_1},$$

$$\Delta = \prod_{\alpha=1}^{\alpha=p-2} (p-\alpha)!\ .\prod_{\beta=1}^{\beta=q-2}(q-\beta)!\ .\ g_1^{\frac{p(p-1)}{2}}\ g_2^{\frac{q(q-1)}{2}}\ (g_2 - g_1)^{pq},$$

$$A_0 = \frac{(\phi - g_2)^q\{k_p(\phi-g_1)^{p-1} + k_{p-1}(\phi-g_1)^{p-2}(g_3-g_1)\ldots\ldots + k_1(g_2-g_1)^{p-1}\}}{(-1)^q(g_2-g_1)^{\omega-1}},$$

$$B_0 = \frac{(\phi - g_1)^p\{h_q(\phi-g_2)^{q-1} + h_{q-1}(\phi-g_2)^{q-2}(g_1-g_2) + \ldots\ldots + h_1(g_1-g_2)^{q-1}\}}{(-1)^{\omega+p-1}(g_2-g_1)^{\omega-1}},$$

where

$$k_1 = 1,$$

$$k_2 = p,$$

$$\vdots$$

$$k_\lambda = \frac{p(p+1)\ \ldots\ldots\ (p+\lambda-2)}{(\lambda-1)!};$$

$$h_1 = 1,$$

$$h_2 = q,$$

$$\vdots$$

$$h_\lambda = \frac{q(q+1)\ \ldots\ldots\ (q+\lambda-2)}{(\lambda-1)!}.$$

6). *Matrix of order* 6.

Latent roots $g_1 = g_4 = g_5$, $g_2 = g_6$, g_3.

$$\Delta = 2!\ g_1^3 g_2 (g_2 - g_1)^6 (g_3 - g_2)^2 (g_1 - g_3)^3,$$

$$\begin{aligned} A_0 = {} & (\phi - g_2)^2 (\phi - g_3) \left[(\phi - g_1)^2 \{ 3(g_3 - g_1)^2 + 2(g_3 - g_1)(g_2 - g_1) + (g_2 - g_1)^2 \} \right. \\ & \left. + (\phi - g_1) \{ 2(g_3 - g_1) + (g_2 - g_1) \} (g_3 - g_1)(g_2 - g_1) + (g_3 - g_1)^2 (g_2 - g_1)^2 \right] \\ & \div (g_2 - g_1)^4 (g_3 - g_1)^3 . \end{aligned}$$

25. *Negative fractional indices.*—We have seen that we may write

$$\phi^{\frac{1}{n}} = A_0 g_1^{\frac{1}{n}} e^{\frac{A_1}{n}} + B_0 g_2^{\frac{1}{n}} e^{\frac{B_1}{n}} + \ldots . + S_0 g_s^{\frac{1}{n}} e^{\frac{S_1}{n}} .$$

Substitute in this $-m$ for n and we have

$$\phi^{-\frac{1}{m}} = A_0 g_1^{-\frac{1}{m}} e^{-\frac{A_1}{m}} + \ldots . + S_0 g_s^{-\frac{1}{m}} e^{-\frac{S_1}{m}},$$

but

$$\phi^{\frac{1}{m}} = A_0 g_1^{\frac{1}{m}} e^{\frac{A_1}{m}} + \ldots . + S_0 g_s^{\frac{1}{m}} e^{\frac{S_1}{m}} .$$

By definition $\phi^{\frac{1}{m}} \phi^{-\frac{1}{m}} = 1$; and multiplying the corresponding sides of these two equations together we have

$$\begin{aligned} \phi^{\frac{1}{m}} \phi^{-\frac{1}{m}} & = A_0 + B_0 + \ldots . + S_0 \\ & = 1 . \end{aligned}$$

In the formula therefore for ϕ^n when no latent root is zero, n may have any integral or fractional positive or negative value.

II. *Some latent roots zero.*—Before proceeding to the case where some but not all the latent roots are zero, I shall consider the case where all the latent roots are zero.

26.—*All latent roots zero—roots of zero.*

In what follows denote the matrix

$$\phi = \left\{ \begin{array}{|cccccc|} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\} \text{ by } 13 + 24 + 35 + 46 .^{*}$$

---

[*] The linear form representation of a matrix is due to Charles S. Peirce; and the notation employed here is virtually his.

The first number of each term of the sum indicating the row and the second the column in which the constituent appears, that is, ϕ is a matrix in which unity is the constituent in the first row and third column, unity is the constituent in the second row and fourth column, unity is the constituent in the third row and fifth column, and unity is the constituent in the fourth row and sixth column, all the other constituents being zero. Similarly in general.

This canonical representation of a matrix was virtually given by Buchheim (in Proc. Lond. Math. Soc. Vol. XVI), but was first explicitly given by Weyr (Comptes Rendus, Vol. C).

If g_α is an α^{uple} latent root of ϕ and if $\alpha_1, \alpha_1+\alpha_2, \ldots \alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_\rho=\alpha$, are the nullities of the matrices $(\phi-g_\alpha), (\phi-g_\alpha)^2 \ldots (\phi-g_\alpha)^\rho$, Weyr terms the numbers $(\alpha, \alpha_1, \alpha_2 \ldots \alpha_\rho)$ the *characteristics* of the latent root g_α.

I shall term two matrices of the same order *equivalent*, if they have the same latent roots with the same characteristics respectively.*

In what follows, since all the latent roots are zero, I shall speak of the characteristics of the matrix instead of the characteristics of the latent root zero.

Consider a matrix ϕ of order ω and suppose

$$N_y[\phi]=p,\ N_y[\phi^2]=p+a_1, \ldots N_y[\phi^t]=p+a_1+a_2+\ldots a_{t-1}=\omega,$$

where $p \geqq a_1 \geqq a_2 > \ldots \geqq a_{t-1}$.

If $\psi^q=\phi$, then

$$N_y[\psi^q]=p,\ N_y[\psi^{2q}]=p+a_1, \ldots N_y[\psi^{qt}]=\omega.$$

Let

$$N_y[\psi]=\alpha, \text{ then}$$

$$q\alpha \geqq p, \text{ and if } \alpha=1,\ q=p$$

$$\text{`` `` } \alpha=2,\ q \geqq \frac{p}{2}.$$

We may now establish the following results, the most of which are restrictive on q, $\frac{1}{q}$ being the index of the root.

1). It is quite obvious that if a_{t-2} is the second last increment of nullity of successive powers of ϕ, then $q \overline{\overline{<}} a_{t-2}$.

---

* Weyr defines two matrices of the same order as being "*matrices de même espèce*" if they have the same latent roots with the same characteristics; and adds that, if M and N are two equivalent matrices, one can always find a matrix Q of nullity zero such that $N=QMQ^{-1}$, which is a formula giving all equivalent matrices in terms of one of them.

2). (a). $\frac{p}{q} \overset{=}{<} \alpha \overset{=}{<} p - q + 1$ when $\frac{p}{q} =$ an integer.

Let $N_y[\psi] = \alpha$,

$$N_y[\psi^2] = \alpha + \alpha_1,$$

$$\vdots$$

$$N_y[\psi^q] = \alpha + \alpha_1 + \ldots\ldots + \alpha_{q-1} = p.$$

Then α will have its least value when $\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_{q-1}$ is greatest, but this sum will be greatest when $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{q-1} = \alpha$;

$\therefore$ α will be least when

$$\alpha + (q-1)\alpha = p,$$

$$\text{or} \quad \alpha = \frac{p}{q}; \quad \therefore \quad \alpha \overset{=}{>} \frac{p}{q}.$$

Again, if α is to have its greatest value, $\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_{q-1}$ must have its least value, but this sum is least when $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{q-1} = 1$;

$\therefore$ α will be greatest when

$$\alpha + q - 1 = p;$$

$$\therefore \quad \alpha = p - q + 1; \quad \therefore \quad \alpha \overset{=}{<} p - q + 1;$$

$\therefore$ $\frac{p}{q} \overset{=}{<} \alpha \overset{=}{<} p - q + 1.$

(b). $\left[\frac{p}{q}\right] + 1 \overset{=}{<} \alpha \overset{=}{<} p - q + 1$ when $\frac{p}{q} \ddagger$ an integer.

Where $\left[\frac{p}{q}\right]$ denotes the greatest integer in $\frac{p}{q}$ as in previous case, α will be least when $\alpha_1 + \alpha_2 + \ldots\ldots + \alpha_{q-1}$ is greatest, that is when $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_{q-2} = \alpha$ and α_{q-1} is as great as possible;

$\therefore$ α is least when

$$\alpha(q-1) + \alpha_{q-1} = p.$$

Now α must be greater than $\left[\frac{p}{q}\right]$; for if $\alpha = \left[\frac{p}{q}\right]$ then $\alpha_{q-1} = p - q\left[\frac{p}{q}\right] + \left[\frac{p}{q}\right]$ which is obviously greater than $\left[\frac{p}{q}\right]$, that is, $\alpha_{q-1} > \alpha$, which is impossible; and therefore $\alpha > \left[\frac{p}{q}\right]$.

Let $\alpha = \left[\frac{p}{q}\right] + 1$;

$$\therefore \quad \alpha_{q-1} = p - q\left[\frac{p}{q}\right] - q + \left[\frac{p}{q}\right] + 1.$$

But $p + 1 - q\left\{\left[\frac{p}{q}\right] + 1\right\} \overset{=}{<} 0$;

$$\therefore \quad \alpha_{q-1} \overset{=}{<} \left[\frac{p}{q}\right] \text{ which is possible;}$$

$$\therefore \quad \alpha = \left[\frac{p}{q}\right] + 1 \text{ is the lower limit.}$$

If α is to have its greatest value, $\alpha_1 + \alpha_2 + \ldots + \alpha_{q-1}$ must have its least value;

$$\therefore \quad \alpha + q - 1 = p;$$
$$\therefore \quad \alpha \overset{=}{<} p - q + 1.$$

Therefore $\left[\frac{p}{q}\right] + 1 \overset{=}{<} \alpha \overset{=}{<} p - q + 1$.

3). If $N_y[\psi^q] = p$ and $q > \frac{p}{2}$, then $N_y[\psi^{q+1}] = p + 1$.

In this case $\alpha \overset{=}{>} 2$ and hence it is obvious that the increments of nullity must reduce to unity at or before the q^{th} power of ψ.

As an immediate consequence of this we have

$$N_y[\psi^{2q}] = p + q,\ N_y[\psi^{3q}] = p + 2q \ldots N_y[\psi^{\kappa q}] = p + (\kappa - 1)q;$$

and $\therefore \quad a_1 = a_2 = \ldots = a_{t-2} = q$.

Therefore there is no q^{th} root of ϕ, q being greater than $\frac{p}{2}$, unless the nullity of successive powers of ϕ increase by equal increments of q.

4). If a_κ is the first increment of nullity that is less than $2q$, then $\psi^{(\kappa-1)q+a_\kappa+1}$ has a nullity $p + 2(\kappa - 2) + 2a_\kappa + 1$, that is, the increment of nullity for all powers of ψ greater than the $\{(\kappa - 1)q + a_\kappa\}^{\text{th}}$ is unity.

$$N_y[\phi^\kappa] = p + 2(\kappa - 1)q;$$
$$\therefore \quad N_y[\psi^{\kappa q}] = p + 2(\kappa - 1)q.$$

Let $\qquad 2x + y = a_\kappa$ and $x + y = q$;

$$\therefore \quad x = a_\kappa - q, \text{ or } y = 2q - a_\kappa.$$
$$N_y[\psi^{\kappa q + x}] = p + 2(\kappa - 1)q + 2x,$$
$$\text{or} \quad N_y[\psi^{\kappa q + a_\kappa - q}] = p + 2(\kappa - 1)q + 2a_\kappa - 2q;$$
$$\therefore \quad N_y[\psi^{(\kappa-1)q + a_\kappa}] = p + 2(\kappa - 2)q + 2a_\kappa.$$

But $$N_y[\psi^{(\varkappa+1)q}] = N_y[\psi^{\varkappa q+\varpi+y}] = p + 2(\varkappa-1)q + a_\varkappa,$$
$$N_y[\psi^{(\varkappa-1)q+a_\varkappa+y}] = p + 2(\varkappa-2) + 2a_\varkappa + y;$$
$$\therefore\quad N_y[\psi^{(\varkappa-1)q+a_\varkappa+1}] = p + 2(\varkappa-2) + 2a_\varkappa + 1;$$
and therefore the increment for all powers of ψ greater than the $\{(\varkappa-1)q+a_\varkappa\}^{\text{th}}$ is unity.

5). If $a_\varkappa$ is the first of the a's that is less than $2q$, then there is no q^{th} root of ϕ unless $a_{\varkappa+1} = a_{\varkappa+2} = \dots\dots a_{t-2} = q$.

This follows as an immediate consequence of 4).

6). If $a_\varkappa$ is the first of the a's that is less than $2q$ and if $a_{\varkappa+1} = a_{\varkappa+2} = \dots\dots = a_{t-2} = q$, then there always exists a q^{th} root of ϕ.

Let $N_y[\psi] = \alpha$, $N_y[\psi^2] = \alpha + 2 \dots\dots N_y[\psi^q] = \alpha + 2(q-1) = p$.

Then
$$\psi = 12 + 23 + \dots\dots(\overline{\varkappa-1}.q+a_\varkappa-1)(\overline{\varkappa-1}.q+a_\varkappa) + * + (\overline{\varkappa-1}.q+a_\varkappa+1)(\overline{\varkappa-1}.q+a_\varkappa+2)$$
$$+ \dots\dots + \dots\dots + (\omega-\alpha+1)(\omega-\alpha+2),$$
where * denotes where a term, which in the natural sequence would appear, has been omitted.

7). If the nullity of successive powers of ϕ increase by equal increments of μq, $(\mu q \leqq p)$, then there is always a q^{th} root of ϕ.

For take
$$\alpha = p - \mu(q-1),$$
and give to successive powers of ψ equal increments of μ;
$$\therefore\quad N_y[\psi^q] = p.$$
And we have
$$\psi_\lambda = \lambda.\overline{\lambda+\mu} + \overline{\lambda+1}.\overline{\lambda+\mu+1} + \dots\dots + \overline{\lambda+\omega-p+\mu(q-1)-1}.\overline{\lambda+\omega-p+\mu(q-1)+\mu-1},$$
where λ may take any of the values
$$1, 2, 3, \dots\dots \overline{p-\mu(q-1)+1}.$$

8). If $q = \frac{p}{2} - \nu$ and $\alpha > 2\nu + 3$, then $N_y[\psi^{q-1}] = p - 1$; and therefore there is no q^{th} root of ϕ unless there are equal increments of q.

Of course $q \geqq 2$; and $\therefore p \geqq 2(\nu + 2)$.

$N_y[\psi] = \alpha$, $N_y[\psi^2] = \alpha + \alpha_1$, etc.;
$$\therefore\quad \alpha + \alpha_1 + \alpha_2 + \dots\dots + \alpha_{q-1} = p.$$

Suppose $\alpha_1 = \alpha_2 = \dots\dots = \alpha_{q-2} = 2$ and $\alpha_{q-1} = 1$;

$$\therefore \quad \alpha + (q-2)2 + 1 = p;$$

$$\therefore \quad \alpha = p - 2q + 3 = 2\nu + 3;$$

and therefore when $\alpha \geqq 2\nu + 3$ obviously $N_y[\psi^{q-1}] = p - 1$.

27. I propose now to show that there is always a q^{th} root of ϕ unless the law of nullity prohibits.

I shall suppose the law of nullity does not prohibit and then show that there is a q^{th} root by finding it.

Suppose $\alpha = i + j + \dots\dots + \varkappa + 1$

and

$$N_y[\psi^{b_0}] = b_0\alpha,\ N_y[\psi^{b_0+b_1}] = b_0\alpha + b_1(\alpha - 1) \dots\dots N_y[\psi^{b_0+b_1+\dots\dots+b_i}] = p,$$

where $q = b_0 + b_1 + b_2 + \dots\dots + b_i$ and

$$p = b_0\alpha + b_1(\alpha - 1) + \dots\dots + b_i(\alpha - i);$$

$$N_y[\psi^{q+b_{i+1}}] = p + b_{i+1}(\alpha - i - 1) \dots\dots N_y[\psi^{q+b_{i+1}+\dots\dots+b_{i+j}}] = p + \alpha_1,$$

where

$$b_{i+1} + b_{i+2} + \dots\dots + b_{i+j} = q \text{ and}$$

$$b_{i+1}(\alpha - i - 1) + \dots\dots + b_{i+j}(\alpha - i - j) = \alpha_1;$$

$N_y[\psi^{2q+b_{i+j+1}}] = p + \alpha_1 + b_{i+j+1}(\alpha - i - j - 1)$; etc.,

etc.,

$$N_y[\psi^{(t-1)q+b_{\alpha-\kappa}+b_{\alpha-\kappa+1}+\dots\dots+b_{\alpha-2}}] = p + \alpha_1 + \alpha_2 + \dots\dots + \alpha_{t-2} + b_{\alpha-2}\cdot 2, \text{ etc.,}$$

$$N_y[\psi^{(t-1)q+b_{\alpha-\kappa}+b_{\alpha-\kappa+1}+\dots\dots+b_{\alpha-1}}] = N_y[\psi^{tq}] = p + \alpha_1 + \alpha_2 + \dots\dots + b_{\alpha-1} = \omega,$$

where $b_{\alpha-1}$ obviously equals α_{t-1}.

We have then for ψ the following:

$$\begin{aligned}
\psi = {} & [12 + 23 + \dots\dots + \overline{b_0 - 1}\,.\,b_0]_{\Sigma b_0} + * + [\overline{b_0 + 1}\,.\,\overline{b_0 + 2} + \dots\dots \\
& \qquad + \overline{2b_0 + b_1 - 1}\,.\,\overline{2b_0 + b_1}]_{\Sigma b_1} + * + \dots\dots \\
& + [\overline{b_0 i + b_1(i-1) + \dots\dots + b_{i-1} + 1}\,.\,\overline{b_0 i + b_1(i-1) + \dots\dots + b_{i-1} + 2} + \dots\dots \\
& \qquad + \overline{b_0(i+1) + \dots\dots + b_i - 1}\,.\,\overline{b_0(i+1) + \dots\dots + b_i}]_{\Sigma b_i} \\
& + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
& + [\overline{b_0(\alpha-1) + b_1(\alpha-2) + \dots\dots b_{\alpha-2} + 1}\,.\,\overline{b_0(\alpha-1) + \dots\dots + b_{i-1} + 2} + \dots\dots \\
& \qquad + \overline{b_0\alpha + \dots\dots + b_{\alpha-1} - 1}\,.\,\overline{b_0\alpha + \dots\dots + b_{\alpha-1}}]_{\Sigma b_{\alpha-1}},
\end{aligned}$$

where Σb_κ denotes $b_0 + b_1 + b_2 + \dots\dots + b_\kappa$ and where $b_0\alpha + b_1(\alpha - 1) + \dots\dots + b_{\alpha-1} = \omega$ the last term being therefore $(\omega - 1)\omega$.

It will be observed that ψ is divided into α sets, each of which I have enclosed in brackets with a subscript indicating the power which causes that set to vanish.

28. I shall now give all possible types of roots of nilpotent matrices of orders 3–10 inclusive.

(a).—*Matrix of order* 3.

$N_y[\phi] = 2$; $N_y[\phi^2] = 3$;

$\phi = 13 = (12 + 23)^2$.

(b).—*Matrix of order* 4.

1). $N_y[\phi] = 2$, $N_y[\phi^2] = 4$;

$\phi = 13 + 24 = (12 + 23 + 34)^2$.

2). $N_y[\phi] = 3$, $N_y[\phi^2] = 4$;

$\phi = 14 = (12 + 23 + 34)^3$;

or $= 24 = (23 + 34)^2$.

(c).—*Matrix of order* 5.

1). $N_y[\phi] = 2$, $N_y[\phi^2] = 4$, $N_y[\phi^3] = 5$;

$\phi = 13 + 24 + 35 = (12 + 23 + 34 + 45)^2$.

2). $N_y[\phi] = 3$, $N_y[\phi^2] = 5$;

$\phi = 14 + 25 = (12 + 23 + 34 + 45)^3$,

or $= 24 + 35 = (23 + 34 + 45)^2$.

3). $N_y[\phi] = 4$, $N_y[\phi^2] = 5$;

$\phi = 15 = (12 + 23 + 34 + 45)^4 = (13 + 24 + 35)^2$,

or $= 25 = (23 + 34 + 45)^3$,

" $= 35 = (34 + 45)^2 = (12 + 34 + 45)^2$.

Here, for the first time thus far, we have more than one type of root of the same index; and hereafter when this occurs I shall give the characteristics.

In this case we have, using *ch.* to denote "characteristics,"

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (5;\ 3,\ 1,\ 1,\ 0) \\ (5;\ 2,\ 2,\ 1,\ 0). \end{cases}$$

(d).—*Matrix of order* 6.

1). $N_y[\phi] = 2$, $N_y[\phi^2) = 4$, $N_y[\phi^3] = 6$.

Instead of indicating the nullity of successive powers of ϕ as heretofore, I shall for convenience simply write the characteristics of ϕ.

In this case we have

ch. $\phi\,(6;\ 2,\ 2,\ 2,\ 0)$;

$\phi = 13 + 24 + 35 + 46 = (12 + 23 + 34 + 45 + 56)^2$.

2). *ch.* $\phi\,(6;\ 3,\ 3,\ 0)$;

$\phi = 14 + 25 + 36 = (12 + 23 + 34 + 45 + 56)^3$,

no. sq. root vide Art. 26. 3).

3). *ch.* $\phi\,(6;\ 3,\ 2,\ 1,\ 0)$;

$\phi = 24 + 35 + 46 = (23 + 34 + 45 + 56)^2$.

4). *ch.* $\phi\,(6;\ 4,\ 2,\ 0)$;

$\phi = 15 + 26 = (12 + 23 + 34 + 45 + 56)^4 = (13 + 24 + 35 + 46)^2$,

or $= 25 + 36 = (23 + 34 + 45 + 56)^3$,

" $= 35 + 46 = (34 + 45 + 56)^2 = (12 + 34 + 45 + 56)^2$,

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (6;\ 3,\ 1,\ 1,\ 1,\ 0) \\ (6;\ 2,\ 2,\ 2,\ 0,) \\ (6;\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

5). *ch.* $\phi\,(6;\ 5,\ 1,\ 0)$;

$\phi = 16 = (12 + 23 + 34 + 45 + 56)^5$,

or $= 26 = (23 + 34 + 45 + 56)^4 = (24 + 35 + 46)^2$,

" $= 36 = (34 + 45 + 56)^3 = (12 + 34 + 45 + 56)^3$,

" $= 46 = (45 + 56)^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (6;\ 4,\ 1,\ 1,\ 0) \\ (6;\ 3,\ 2,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (6;\ 3,\ 1,\ 1,\ 1,\ 0) \\ (6;\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

(e).—*Matrix of order* 7.

1). *ch.* $\phi\,(7;\ 2,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 13 + 24 + 35 + 46 + 57 = (12 + 23 + 34 + \text{etc.})^2$.

2). *ch.* $\phi\,(7;\ 3,\ 3,\ 1,\ 0)$;

$\phi = 14 + 25 + 36 + 47 = (12 + 23 + 34 + \text{etc.})^3$,

no. sq. root.

3). *ch.* $\phi\,(7;\ 3,\ 2,\ 2,\ 0)$;

$\phi = 24 + 35 + 46 + 57 = (23 + 34 + 45 + 56 + 67)^2$.

4). *ch.* $\phi\,(7;\ 4,\ 3,\ 0)$;

$\phi = 15 + 26 + 37 = (12 + 23 + 34 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$,

or $= 25 + 36 + 47 = (23 + 34 + \text{etc.})^3$.

5). *ch.* $\phi\,(7;\ 4,\ 2,\ 1,\ 0)$;

$$\phi = 35 + 46 + 57 = (34 + 45 + 56 + 67)^2,$$
$$= (12 + 34 + 45 + 56 + 67)^2,$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

6). *ch.* $\phi\,(7;\ 5,\ 2,\ 0)$;

$$\phi = 16 + 27 = (12 + 23 + \text{etc.})^5,$$
$$\text{or} = 26 + 37 = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2,$$
$$\text{“} = 36 + 47 = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3,$$
$$\text{“} = 46 + 57 = (45 + 56 + 67)^2 = (12 + 45 + 56 + 67)^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7;\ 4,\ 1,\ 1,\ 1,\ 0) \\ (7;\ 3,\ 2,\ 2,\ 0) \\ (7;\ 3,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (7;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

7). *ch.* $\phi\,(7;\ 6,\ 1,\ 0)$;

$$\phi = 17 = (12 + 23 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3 = (14 + 25 + \text{etc.})^2,$$
$$\text{or} = 27 = (23 + 34 + \text{etc.})^5,$$
$$\text{“} = 37 = (34 + 45 + 56 + 67)^4 = (12 + 34 + 45 + \text{etc.})^4$$
$$= (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + 57)^2,$$
$$\text{“} = 47 = (45 + 56 + 67)^3 = (12 + 45 + 56 + 67)^3$$
$$= (12 + 23 + 45 + 56 + 67)^3,$$
$$\text{“} = 57 = (56 + 67)^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (7;\ 5,\ 1,\ 1,\ 0) \\ (7;\ 4,\ 2,\ 1,\ 0) \\ (7;\ 3,\ 3,\ 1,\ 0), \end{cases} \quad ch.\ \phi^{\frac{1}{3}} \begin{cases} (7;\ 4,\ 1,\ 1,\ 1,\ 0) \\ (7;\ 3,\ 2,\ 1,\ 1,\ 0) \\ (7;\ 2,\ 2,\ 2,\ 1,\ 0), \end{cases} \quad ch.\ \phi^{\frac{1}{4}} \begin{cases} (7;\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (7;\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

(f). *Matrix of order* 8.

1). *ch.* $\phi\,(8;\ 2,\ 2,\ 2,\ 2,\ 0)$;

$$\phi = 13 + 24 + \text{etc.} = (12 + 23 + \text{etc.})^2.$$

2). *ch.* $\phi\,(8;\ 3,\ 3,\ 2,\ 0)$;

$$\phi = 14 + 25 + \text{etc.} = (12 + 23 + \text{etc.})^3.$$

3). *ch.* $\phi\,(8;\ 3,\ 2,\ 2,\ 1,\ 0)$;

$$\phi = 24 + 35 + 46 + 57 + 68 = (23 + 34 + \text{etc.})^2.$$

4). *ch.* $\phi\,(8;\ 4,\ 4,\ 0)$;

$$\phi = 15 + 26 + 37 + 48 = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2,$$

no. cu. root.

5). $ch.\ \phi\ (8\,;\ 4,\ 3,\ 1,\ 0)$;

$$\phi = 25 + 36 + 47 + 58 = (23 + 34 + 45 + \text{etc.})^3,$$
$$\text{or} = 13 + 46 + 57 + 68 = (12 + 23 + 45 + 56 + 67 + 78)^2.$$

6). $ch.\ \phi\ (8\,;\ 4,\ 2,\ 2,\ 0)$;

$$\phi = 35+46+57+68 = (34+45+56+\text{etc.})^2 = (12+34+45+56+\text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

7). $ch.\ \phi\ (8\,;\ 5,\ 3,\ 0)$;

$$\phi = 16 + 27 + 38 = (12 + 23 + \text{etc.})^5,$$
$$\text{or} = 26 + 37 + 48 = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2,$$
$$\text{``} = 36 + 47 + 58 = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3.$$

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

8). $ch.\ \phi\ (8\,;\ 5,\ 2,\ 1,\ 0)$;

$$\phi = 46 + 57 + 68 = (45 + 56 + 67 + 78)^2 = (12+45+56+67+78)^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

9). $ch.\ \phi\ (8\,;\ 6,\ 2,\ 0)$;

$$\begin{aligned} \phi = 17 + 28 &= (12 + 23 + 34 + \text{etc.})^6 = (13 + 24 + 35 + \text{etc.})^3 \\ &= (14 + 25 + \text{etc.})^2, \\ \text{or} = 27 + 38 &= (23 + 34 + \text{etc.})^5, \\ \text{``} = 37 + 48 &= (34 + 45 + 56 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4 \\ &= (35 + 46 + \text{etc.})^2, \\ \text{``} = 47 + 58 &= (45 + 56 + \text{etc.})^3 = (12 + 45 + 56 + \text{etc.})^3 \\ &= (12 + 23 + 45 + 56 + \text{etc.})^3, \\ \text{``} = 57 + 68 &= (56 + 67 + 78)^2 = (12 + 56 + \text{etc.})^2 \\ &= (12 + 34 + 56 + \text{etc.})^2. \end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 5,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 4,\ 2,\ 2,\ 0) \\ (8\,;\ 4,\ 2,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 3,\ 2,\ 0) \\ (8\,;\ 3,\ 3,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 2,\ 2,\ 0) \\ (8\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

10). *ch.* $\phi\ (8\,;\ 7,\ 1,\ 0)$;

$\phi = 18 = (12 + 23 + \text{etc.})^7$,

or $= 28 = (23 + 34 + \text{etc.})^6 = (24 + 35 + \text{etc.})^3 = (25 + 36 + \text{etc.})^2$,

" $= 38 = (34 + 45 + \text{etc.})^5 = (12 + 34 + 45 + \text{etc.})^5$,

" $= 48 = (45 + 56 + 67 + 78)^4 = (12 + 45 + \text{etc.})^4$

$= (12 + 23 + 45 + \text{etc.})^4 = (46 + 57 + 68)^2$

$= (12 + 46 + 57 + 68)^2$,

" $= 58 = (56 + 67 + 78)^3 = (12 + 56 + 67 + 78)^3$

$= (12 + 34 + 56 + 67 + 78)^3$,

" $= 68 = (67 + 78)^2 = (12 + 67 + 78)^2 = (12 + 34 + 67 + 78)^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (8\,;\ 6,\ 1,\ 1,\ 0) \\ (8\,;\ 5,\ 2,\ 1,\ 0) \\ (8\,;\ 4,\ 3,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (8\,;\ 5,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 4,\ 2,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 3,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 2,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (8\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{5}} \begin{cases} (8\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (8\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

(g).—*Matrix of order* 9.

1). *ch.* $\phi\ (9\,;\ 2,\ 2,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 13 + 24 + \text{etc.} = (12 + 23 + \text{etc.})^2$.

2). *ch.* $\phi\ (9\,;\ 3,\ 3,\ 3,\ 0)$;

$\phi = 14 + 25 + 36 + \text{etc.} = (12 + 23 + \text{etc.})^3$.

3). *ch.* $\phi\ (9\,;\ 3,\ 2,\ 2,\ 2,\ 0)$;

$\phi = 24 + 35 + 46 + \text{etc.} = (23 + 34 + \text{etc.})^2$.

4). *ch.* $\phi\ (9\,;\ 4,\ 4,\ 1,\ 0)$;

$\phi = 15 + 26 + 37 + 48 + 59 = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$.

5). *ch.* $\phi\ (9\,;\ 4,\ 3,\ 2,\ 0)$;

$\phi = 25 + 36 + \text{etc.} = (23 + 34 + \text{etc.})^3$,

or $= 13 + 46 + 57 + 68 + 79 = (12 + 23 + 45 + 56 + \text{etc.})^2$.

6). *ch.* $\phi\ (9\,;\ 4,\ 2,\ 2,\ 1,\ 0)$;

$\phi = 35 + 46 + \text{etc.} = (34 + 45 + \text{etc.})^2 = (12 + 34 + 45 + \text{etc.})^2$.

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (9\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

7). *ch.* $\phi\,(9;\ 5,\ 4,\ 0)$;

$$\phi = 16 + 27 + 38 + 49 = (12 + 23 + \text{etc.})^5,$$

$$\text{or} = 26+37+48+59 = (23+34+45+\text{etc.})^4 = (24+35+46+\text{etc.})^2.$$

8). *ch.* $\phi\,(9;\ 5,\ 3,\ 1,\ 0)$;

$$\phi = 36 + 47 + 58 + 69 = (34 + 45 + \text{etc.})^3 = (12 + 34 + \text{etc.})^3,$$

$$\text{or} = 13 + 57 + 68 + 79 = (12 + 23 + 57 + \text{etc.})^3.$$

$$ch.\ \phi^{\frac{1}{3}} \left\{ \begin{array}{l} (9;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{array} \right.$$

9). *ch.* $\phi\,(9;\ 5,\ 2,\ 2,\ 0)$;

$$\phi = 46 + 57 + 68 + 79 = (45 + 56 + \text{etc.})^2 = (12 + 45 + 56 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \left\{ \begin{array}{l} (9;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{array} \right.$$

10). *ch.* $\phi\,(9;\ 6,\ 3,\ 0)$;

$$\begin{aligned} \phi = 17 + 28 + 39 &= (12 + 23 + 34 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3 \\ &= (14 + 25 + 36 + \text{etc.})^2, \\ \text{or} = 27 + 38 + 49 &= (23 + 34 + \text{etc.})^5, \\ \text{``} = 37 + 48 + 59 &= (34 + 45 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4 \\ &= (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + \text{etc.})^2, \\ \text{``} = 47 + 58 + 69 &= (45 + 56 + \text{etc.})^3 = (12 + 45 + 56 + \text{etc.})^3 \\ &= (12 + 23 + 45 + 56 + \text{etc.})^3. \end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \left\{ \begin{array}{l} (9;\ 4,\ 2,\ 2,\ 1,\ 0) \\ (9;\ 3,\ 3,\ 3,\ 0) \\ (9;\ 3,\ 3,\ 2,\ 1,\ 0), \end{array} \right. \qquad ch.\ \phi^{\frac{1}{3}} \left\{ \begin{array}{l} (9;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0), \end{array} \right.$$

$$ch.\ \phi^{\frac{1}{4}} \left\{ \begin{array}{l} (9;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{array} \right.$$

11). *ch.* $\phi\,(9;\ 6,\ 2,\ 1,\ 0)$;

$$\begin{aligned} \phi = 57 + 68 + 79 &= (56 + 67 + \text{etc.})^2 = (12 + 56 + 67 + \text{etc.})^2 \\ &= (12 + 34 + 56 + 67 + \text{etc.})^2. \end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \left\{ \begin{array}{l} (9;\ 5,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 4,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 3,\ 1,\ 1,\ 1,\ 0). \end{array} \right.$$

12). *ch.* $\phi\,(9;\ 7,\ 2,\ 0)$;

$$\begin{aligned}
\phi &= 18 + 29 = (12 + 23 + \text{etc.})^7, \\
\text{or } &= 28 + 39 = (23 + 34 + \text{etc.})^6 = (24 + 35 + \text{etc.})^3 \\
&\qquad\qquad = (25 + 36 + \text{etc.})^2, \\
\text{“} &= 38 + 49 = (34 + 45 + \text{etc.})^5 = (12 + 34 + 45 + \text{etc.})^5, \\
\text{“} &= 48 + 59 = (45 + 56 + \text{etc.})^4 = (12 + 45 + 56 + \text{etc.})^4 \\
&\qquad\qquad = (12 + 23 + 45 + 56 + \text{etc.})^4, \\
\text{“} &= 58 + 69 = (56 + 67 + \text{etc.})^3 = (12 + 56 + 67 + \text{etc.})^3 \\
&\qquad\qquad = (12 + 34 + 56 + \text{etc.})^3 = (12 + 23 + 56 + \text{etc.})^3, \\
\text{“} &= 67 + 79 = (67 + 78 + 89)^2 = (12 + 67 + 78 + 89)^2 \\
&\qquad\qquad = (12 + 34 + 67 + \text{etc.})^2, \\
\text{“} &= 13 + 79 = (12 + 23 + 78 + 89)^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (9;\ 6,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 5,\ 2,\ 2,\ 0) \\ (9;\ 5,\ 2,\ 1,\ 1,\ 0) \\ (9;\ 4,\ 3,\ 2,\ 0) \\ (9;\ 4,\ 3,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (9;\ 5,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 4,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 3,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 2,\ 2,\ 0) \\ (9;\ 3,\ 2,\ 2,\ 1,\ 1,\ 0), \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (9;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{5}} \begin{cases} (9;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (9;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

13). *ch.* $\phi\,(9;\ 8,\ 1,\ 0)$;

$$\begin{aligned}
\phi &= 19 = (12 + 23 + \text{etc.})^8 = (13 + 24 + \text{etc.})^4 = (15 + 26 + \text{etc.})^2, \\
\text{or } &= 29 = (23 + 34 + \text{etc.})^7, \\
\text{“} &= 39 = (34 + 45 + 56 + \text{etc.})^6 = (12 + 34 + \text{etc.})^6 \\
&\qquad = (35 + 46 + \text{etc.})^3 = (12 + 35 + 46 + \text{etc.})^3 \\
&\qquad = (36 + 47 + \text{etc.})^2 = (12 + 36 + 47 + \text{etc.})^2, \\
\text{“} &= 49 = (45 + 56 + \text{etc.})^5 = (12 + 45 + 56 + \text{etc.})^5 \\
&\qquad = (12 + 23 + 45 + \text{etc.})^5, \\
\text{“} &= 59 = (56 + 67 + \text{etc.})^4 = (12 + 56 + 67 + \text{etc.})^4 \\
&\qquad = (12 + 23 + 56 + 67 + \text{etc.})^4 = (12 + 34 + 56 + 67 + \text{etc.})^4 \\
&\qquad = (12 + 23 + 34 + 56 + 67 + \text{etc.})^4 = (57 + 68 + \text{etc.})^2 \\
&\qquad = (12 + 57 + 68 + 79)^2 = (12 + 34 + 57 + \text{etc.})^2, \\
\text{“} &= 69 = (67 + 78 + 89)^3 = (12 + 67 + 78 + 89)^3 \\
&\qquad = (12 + 34 + 67 + \text{etc.})^3 = (12 + 23 + 67 + 78 + 89)^3 \\
&\qquad = (12 + 23 + 45 + \text{etc.})^3, \\
\text{“} &= 79 = (78 + 89)^2 = (12 + 78 + 89)^2.
\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\begin{cases}(9\,;\ 7,\ 1,\ 1,\ 0)\\(9\,;\ 6,\ 2,\ 1,\ 0)\\(9\,;\ 5,\ 3,\ 1,\ 0)\\(9\,;\ 4,\ 4,\ 1,\ 0),\end{cases}\qquad ch.\ \phi^{\frac{1}{3}}\begin{cases}(9\,;\ 6,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 5,\ 2,\ 1,\ 1,\ 0)\\(9\,;\ 4,\ 3,\ 1,\ 1,\ 0)\\(9\,;\ 4,\ 2,\ 2,\ 1,\ 0)\\(9\,;\ 3,\ 3,\ 2,\ 1,\ 0),\end{cases}$$

$$ch.\ \phi^{\frac{1}{4}}\begin{cases}(9\,;\ 5,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 4,\ 2,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 3,\ 3,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 2,\ 2,\ 2,\ 2,\ 1,\ 0),\end{cases}\qquad ch.\ \phi^{\frac{1}{5}}\begin{cases}(9\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0),\end{cases}$$

$$ch.\ \phi^{\frac{1}{6}}\begin{cases}(9\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(9\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

(h).—*Matrix of order* 10.

1). $ch.\ \phi\ (10\,;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0)$;
$\phi = 13 + 24 + 35 + \text{etc.} = (12 + 23 + \text{etc.})^2$.

2). $ch.\ \phi\ (10\,;\ 3,\ 3,\ 3,\ 1,\ 0)$;
$\phi = 14 + 25 + \text{etc.} = (12 + 23 + \text{etc.})^3$.

3). $ch.\ \phi\ (10\,;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0)$;
$\phi = 24 + 35 + \text{etc.} = (23 + 34 + \text{etc.})^2$.

4). $ch.\ \phi\ (10\,;\ 4,\ 4,\ 2,\ 0)$;
$\phi = 15 + 26 + \text{etc.} = (12 + 23 + \text{etc.})^4 = (13 + 24 + \text{etc.})^2$.

5). $ch.\ \phi\ (10\,;\ 4,\ 3,\ 3,\ 0)$;
$\phi = 25 + 36 + \text{etc.} = (23 + 34 + \text{etc.})^3$.

6). $ch.\ \phi\ (10\,;\ 4,\ 3,\ 2,\ 1,\ 0)$;
$\phi = 13 + 46 + 57 + \text{etc.} = (12 + 23 + \text{etc.})^3$.

7). $ch.\ \phi\ (10\,;\ 4,\ 2,\ 2,\ 2,\ 0)$;

$$\begin{aligned}\phi = 35 + 46 + 57 + \text{etc.}) &= (34 + 45 + \text{etc.})^2\\ &= (12 + 34 + 45 + \text{etc.})^2.\end{aligned}$$

$$ch.\ \phi^{\frac{1}{2}}\begin{cases}(10\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0)\\(10\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0).\end{cases}$$

8). $ch.\ \phi\ (10\,;\ 5,\ 5,\ 0)$;
$\phi = 16 + 27 + \text{etc.} = (12 + 23 + \text{etc.})^5$.

9). $ch.\ \phi\ (10\,;\ 5,\ 4,\ 1,\ 0)$;
$\phi = 26 + 37 + \text{etc.} = (23 + 34 + \text{etc.})^4 = (24 + 35 + \text{etc.})^2$.

10). $ch.\ \phi\ (10;\ 5,\ 3,\ 2,\ 0)$;

$$\phi = 36 + 47 + \text{etc.} = (34 + 45 + \text{etc.})^3 = (12 + 34 + 45 + \text{etc.})^3,$$
$$\text{or} = 13 + 57 + \text{etc.} = (12 + 23 + 56 + 67 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

11). $ch.\ \phi\ (10;\ 5,\ 2,\ 2,\ 1,\ 0)$;

$$\phi = 46 + 57 + 68 + \text{etc.} = (45 + 56 + \text{etc.})^2 = (12 + 45 + 56 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

12). $ch.\ \phi\ (10;\ 6,\ 4,\ 0)$;

$$\phi = 17 + 28 + \text{etc.} = (12 + 23 + \text{etc.})^6 = (13 + 24 + \text{etc.})^3$$
$$= (14 + 25 + \text{etc.})^2,$$
$$\text{or} = 27 + 38 + \text{etc.} = (23 + 34 + \text{etc.})^5,$$
$$\text{“} = 37 + 48 + \text{etc.} = (34 + 45 + \text{etc.})^4 = (12 + 34 + 45 + \text{etc.})^4$$
$$= (35 + 46 + \text{etc.})^2 = (12 + 35 + 46 + \text{etc.})^2,$$
$$\text{“} = 14 + 58 + 69 + 710 = (12 + 23 + 34 + 56 + 67 + \text{etc.})^3.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 3,\ 3,\ 3,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 2,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0). \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

13). $ch.\ \phi\ (10;\ 6,\ 3,\ 1,\ 0)$;

$$\phi = 47 + 58 + \text{etc.} = (45 + 56 + \text{etc.})^3 = (12 + 45 + \text{etc.})^3$$
$$= (12 + 23 + 45 + \text{etc.})^3,$$
$$\text{or} = 13 + 68 + 79 + 810 = (12 + 23 + 67 + 78 + \text{etc.})^2$$
$$= (12 + 23 + 45 + 67 + 78 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

14). $ch.\ \phi\ (10;\ 6,\ 2,\ 2,\ 0)$;

$$\phi = 57 + 68 + 79 + 810 = (56 + 67 + \text{etc.})^2 = (12 + 56 + \text{etc.})^2$$
$$= (12 + 34 + 56 + \text{etc.})^2.$$

$$ch.\ \phi^{\frac{1}{3}} \begin{cases} (10\,;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

15). $ch.\ \phi\ (10\,;\ 7,\ 3,\ 0)$;

$\phi = 18 + 29 + 310 = (12 + 23 + \text{etc.})^7$,

or $= 28 + 39 + 410 = (23 + 34 + \text{etc.})^6 = (24 + 35 + \text{etc.})^3$

$= (25 + 36 + \text{etc.})^2$,

" $= 38 + 49 + 510 = (34 + 45 + \text{etc.})^5 = (12 + 34 + 45 + \text{etc.})^5$,

" $= 48 + 59 + 610 = (45 + 56 + \text{etc.})^4 = (12 + 45 + 56 + \text{etc.})^4$

$= (12 + 23 + 45 + \text{etc.})^4 = (46 + 57 + \text{etc.})^2$

$= (12 + 46 + 57 + \text{etc.})^2$,

" $= 58 + 69 + 710 = (56 + 67 + \text{etc.})^3 = (12 + 56 + 67 + \text{etc.})^3$

$= (12 + 23 + 56 + \text{etc.})^3 = (12 + 34 + 56 + \text{etc.})^3$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10\,;\ 5,\ 2,\ 2,\ 1,\ 0) \\ (10\,;\ 4,\ 3,\ 3,\ 0) \\ (10\,;\ 4,\ 3,\ 2,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{3}} \begin{cases} (10\,;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0) \\ (10\,;\ 3,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

$$ch.\ \phi^{\frac{1}{4}} \begin{cases} (10\,;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0), \end{cases} \qquad ch.\ \phi^{\frac{1}{5}} \begin{cases} (10\,;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

16). $ch.\ \phi\ (10\,;\ 7,\ 2,\ 1,\ 0)$;

$\phi = 68 + 79 + 810 = (67 + 78 + \text{etc.})^2 = (12 + 67 + 78 + \text{etc.})^2$

$= (12 + 34 + 67 + 78 + \text{etc.})^2$.

$$ch.\ \phi^{\frac{1}{2}} \begin{cases} (10\,;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10\,;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0). \end{cases}$$

17). $ch.\ \phi\ (10\,;\ 8,\ 2,\ 0)$;

$\phi = 19 + 210 = (12 + 23 + \text{etc.})^8 = (13 + 24 + \text{etc.})^4$

$= (15 + 26 + \text{etc.})^2$,

or $= 29 + 310 = (23 + 34 + \text{etc.})^7$,

" $= 39 + 410 = (34 + 45 + \text{etc.})^6 = (12 + 34 + 45 + \text{etc.})^6$

$= (35 + 46 + \text{etc.})^3 = (12 + 35 + \text{etc.})^3$

$= (36 + 47 + \text{etc.})^2 = (12 + 36 + 47 + \text{etc.})^2$,

or $= 49 + 510 = (45 + 56 + 67 + \text{etc.})^5 = (12 + 45 + 56 + \text{etc.})^5$
$ = (12 + 23 + 45 + \text{etc.})^5,$
" $= 59 + 610 = (56 + 67 + \text{etc.})^4 = (12 + 56 + 67 + \text{etc.})^4$
$= (12 + 23 + 56 + \text{etc.})^4 = (12 + 34 + 56 + \text{etc.})^4$
$= (12 + 23 + 34 + 56 + \text{etc.})^4 = (57 + 68 + \text{etc.})^2$
$= (12 + 57 + 68 + \text{etc.})^2 = (12 + 34 + 57 + \text{etc.})^2,$
" $= 69 + 710 = (67 + 78 + \text{etc.})^3 = (12 + 67 + 78 + \text{etc.})^3$
$= (12 + 23 + 67 + 78 + \text{etc.})^3$
$= (12 + 23 + 45 + 67 + 78 + \text{etc.})^3.$
$= (12 + 34 + 67 + 78 + \text{etc.})^3,$
" $= 79 + 810 = (78 + 89 + 910)^2 = (12 + 78 + 89 + 910)^2$
$= (12 + 34 + 78 + \text{etc.})^2 = (12+34+56+78+ \text{etc.})^2.$

$$ch.\ \phi^{\frac{1}{2}} \left\{ \begin{array}{l} (10;\ 7,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 6,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 6,\ 2,\ 2,\ 0) \\ (10;\ 5,\ 3,\ 2,\ 0) \\ (10;\ 5,\ 3,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 4,\ 2,\ 0) \\ (10;\ 4,\ 4,\ 1,\ 1,\ 0), \end{array} \right. \qquad ch.\ \phi^{\frac{1}{3}} \left\{ \begin{array}{l} (10;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 2,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0), \end{array} \right.$$

$$ch.\ \phi^{\frac{1}{4}} \left\{ \begin{array}{l} (10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 2,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{array} \right. \qquad \begin{array}{l} ch.\ \phi^{\frac{1}{5}} \left\{ \begin{array}{l} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0), \end{array} \right. \\ \\ ch.\ \phi^{\frac{1}{6}} \left\{ \begin{array}{l} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{array} \right. \end{array}$$

18). *ch.* $\phi\ (10;\ 9,\ 1,\ 0)$;
$\phi = 110 = (12 + 23 + \text{etc.})^9 = (14 + 25 + \text{etc.})^3,$
or $= 210 = (23 + 34 + \text{etc.})^8 = (24 + 35 + \text{etc.})^4$
$= (26 + 37 + \text{etc.})^2,$
" $= 310 = (34 + 45 + \text{etc.})^7 = (12 + 34 + 45 + \text{etc.})^7,$
" $= 410 = (45 + 56 + \text{etc.})^6 = (12 + 45 + \text{etc.})^6$
$= (12 + 23 + 45 + 56 + \text{etc.})^6 = (46 + 57 + \text{etc.})^3$
$= (12 + 46 + 57 + \text{etc.})^3 = (12 + 23 + 46 + \text{etc.})^3$
$= (47 + 58 + \text{etc.})^2 = (12 + 47 + \text{etc.})^2 = \text{etc.},$

$$
\begin{aligned}
\text{or } = 510 &= (56 + 67 + \text{etc.})^5 = (12 + 56 + \text{etc.})^5 \\
&= (12 + 23 + 56 + \text{etc.})^5 = (12 + 34 + 56 + \text{etc.})^5 \\
&= (12 + 23 + 34 + 56 + \text{etc.})^5, \\
\text{“} = 610 &= (67 + 78 + \text{etc.})^4 = (12 + 67 + 78 + \text{etc.})^4 \\
&= (12 + 34 + 67 + \text{etc.})^4 = (12 + 23 + 67 + 78 + \text{etc.})^4 \\
&= (12 + 23 + 34 + 67 + 78 + \text{etc.})^4 \\
&= (12 + 23 + 45 + 67 + \text{etc.})^4 = (68 + 79 + 810)^2 \\
&= (12 + 68 + 79 + \text{etc.})^2 = (12 + 34 + 68 + \text{etc.})^2, \\
\text{“} = 710 &= (78 + 89 + 910)^3 = (12 + 78 + \text{etc.})^3 \\
&= (12 + 23 + 78 + \text{etc.})^3 = (12 + 34 + 56 + 78 + \text{etc.})^3 \\
&= (12 + 23 + 45 + 78 \text{ etc.})^3 \\
&= (12 + 23 + 45 + 56 + 78 + \text{etc.})^3 \\
&= (12 + 34 + 56 + 78 + \text{etc.})^3, \\
\text{“} = 810 &= (89 + 910)^2 = (12 + 89 + 910)^2 \\
&= (12 + 34 + 89 + 910)^2 = (12 + 34 + 56 + 89 + 910)^2.
\end{aligned}
$$

$$
ch.\ \phi^{\frac{1}{4}} \left\{ \begin{array}{l} (10;\ 8,\ 1,\ 1,\ 0) \\ (10;\ 7,\ 2,\ 1,\ 0) \\ (10;\ 6,\ 3,\ 1,\ 0) \\ (10;\ 5,\ 4,\ 1,\ 0), \end{array} \right. \qquad
ch.\ \phi^{\frac{1}{5}} \left\{ \begin{array}{l} (10;\ 7,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 6,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 5,\ 3,\ 1,\ 1,\ 0) \\ (10;\ 5,\ 2,\ 2,\ 1,\ 0) \\ (10;\ 4,\ 4,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 3,\ 2,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 3,\ 1,\ 0), \end{array} \right.
$$

$$
ch.\ \phi^{\frac{1}{6}} \left\{ \begin{array}{l} (10;\ 6,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 5,\ 2,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 3,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 2,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 2,\ 2,\ 1,\ 0), \end{array} \right. \qquad
ch.\ \phi^{\frac{1}{7}} \left\{ \begin{array}{l} (10;\ 5,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 4,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 3,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 0), \end{array} \right.
$$

$$
ch.\ \phi^{\frac{1}{8}} \left\{ \begin{array}{l} (10;\ 4,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 3,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 0), \end{array} \right. \qquad
ch.\ \phi^{\frac{1}{9}} \left\{ \begin{array}{l} (10;\ 3,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0) \\ (10;\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 0). \end{array} \right.
$$

29. Proceeding now to the consideration of matrices having some latent roots zero and others different from zero it may be observed that:

(1). The nullity of N depends on the nullity of ϕ and in the following way:

$$N_y[N^{\nu}] = N_y[\phi^{\nu}] + \omega - p_s .$$

And consequently when the nullity of ϕ is equal to its vacuity the nullity of N is ω, and therefore N vanishes.

(2). When the nullity of ϕ is equal to its vacuity it has an n^{th} root, and when the nullity is less than the vacuity ϕ obviously cannot have a root with index greater than its nullity.

(3). If $N_y[\phi] = p$, ϕ cannot have a $(p-\mu)^{\text{th}}$ root unless N has a $(p-\mu)^{\text{th}}$ root.

30. N is a nilpotent matrix—a root of zero—such as was considered in Art. 28, and as was there shown will have a q^{th} root unless the law of nullity prohibits. The relation existing between the nullities of N and ϕ shows us that if the law of nullity permits one it will also permit the other to have a q^{th} root, and consequently we have the theorem that: There will always be a q^{th} root unless the law of nullity prohibits.

§6.—*Transcendental Functions of a Matrix.*

31. In this section I shall consider a few cases of the elementary transcendental functions of a matrix.

(a). *Exponential function.*—I define e^{ϕ} by the ordinary series, viz.:

$$e^{\phi} = 1 + \frac{\phi}{1} + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \text{etc.} + \frac{\phi^n}{n!} + \text{etc.}$$

$$= \sum_0^{\infty}{}_{\mu} \frac{\phi^{\mu}}{\mu!}.$$

Let $\phi^n = (A_0 + nA_1 + n^2A_2 + \ldots\ldots + n^{p_1}A_{p_1})g_1^n + (B_0 + nB_1 + \ldots\ldots + n^{p_2}B_{p_2})g_2^n + \ldots\ldots + (S_0 + nS_1 + n^2S_2 + \ldots\ldots + n^{p_s}S_{p_s})g_s^n$,

then

$$e^{\phi} = \sum_0^{\infty}{}_{\mu} \left\{ \frac{(A_0 + \mu A_1 + \ldots\ldots + \mu^{p_1}A_{p_1})g_1^{\mu} + \ldots\ldots + (S_0 + \mu S_1 + \ldots\ldots + \mu^{p_s}S_{p_s})g_s^{\mu}}{\mu!} \right\}$$

$$
\begin{aligned}
&= \sum_{0}^{\infty}{}^{\mu} A_0 \frac{g_1^{\mu}}{\mu!} + \sum_{0}^{\infty}{}^{\mu} \mu A_1 \frac{g_1^{\mu}}{\mu!} + \sum_{0}^{\infty}{}^{\mu} \mu^2 A_2 \frac{g_1^{\mu}}{\mu!} + \ldots . + \sum_{0}^{\infty}{}^{\mu} \mu^{p_1} A_{p_1} \frac{g_1^{\mu}}{\mu!} \\
&\quad + \sum_{0}^{\infty}{}^{\mu} B_0 \frac{g_2^{\mu}}{\mu!} + \sum_{0}^{\infty}{}^{\mu} \mu B_1 \frac{g_2^{\mu}}{\mu!} + \ldots . + \sum_{0}^{\infty}{}^{\mu} \mu^{p_2} B_{p_2} \frac{g_2^{\mu}}{\mu!} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \sum_{0}^{\infty}{}^{\mu} S_0 \frac{g_s^{\mu}}{\mu!} + \sum_{0}^{\infty}{}^{\mu} \mu S_1 \frac{g_s^{\mu}}{\mu!} + \ldots . + \sum_{0}^{\infty}{}^{\mu} \mu^{p_s} S_{p_s} \frac{g_s^{\mu}}{\mu!} \\
&= A_0 e^{g_1} + g_1 A_1 e^{g_1} + \ldots . + \sum_{1}^{p_1}{}^{\lambda} g_1^{\lambda} A_{p_1} e^{g_1} \\
&\quad + B_0 e^{g_2} + g_2 B_1 e^{g_2} + \ldots . + \sum_{1}^{p_2}{}^{\lambda} g_2^{\lambda} B_{p_2} e^{g_2} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + S_0 e^{g_s} + g_s S_1 e^{g_s} + \ldots . + \sum_{1}^{p_s}{}^{\lambda} g_s^{\lambda} S_{p_s} e^{g_s}.
\end{aligned}
$$

Again

$$
\begin{aligned}
e^{-\phi} &= \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{\phi^{\mu}}{\mu!} \\
&= \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \left\{ \frac{(A_0 + \mu A_1 + \ldots + \mu^{p_1} A_{p_1}) g_1^{\mu} + \ldots + (S_0 + \mu S_1 + \ldots + \mu^{p_s} S_{p_s}) g_s^{\mu}}{\mu!} \right\} \\
&= A_0 \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{g_1^{\mu}}{\mu!} + \ldots . + \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \mu^{p_1} A_{p_1} \frac{g_1^{\mu}}{\mu!} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + S_0 \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \frac{g_s^{\mu}}{\mu!} + \ldots . + \sum_{0}^{\infty}{}^{\mu} (-1)^{\mu+2} \mu^{p_s} S_{p_s} \frac{g_s^{\mu}}{\mu!}, \\
&= \{ A_0 - g_1 A_1 - (g_1 - g_1^2) A_2 - \ldots . - \sum_{1}^{p_1}{}^{\mu} (-1)^{\mu+1} g_1^{\mu} A_{p_1} \} e^{-g_1} \\
&\quad + \{ B_0 - g_2 B_1 - (g_2 - g_2^2) B_2 - \ldots . - \sum_{1}^{p_2}{}^{\mu} (-1)^{\mu+1} g_2^{\mu} B_{p_2} \} e^{-g_2} \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + \{ S_0 - g_s S_1 - (g_s - g_s^2) S_2 - \ldots . - \sum_{1}^{p_s}{}^{\mu} (-1)^{\mu+1} g_s^{\mu} S_{p_s} \} e^{-g_s}.
\end{aligned}
$$

(b). *Logarithmic function.*—I define $\log\phi$ by the series:

$$\log\phi = (\phi-1) - \tfrac{1}{2}(\phi-1)^2 + \tfrac{1}{3}(\phi-1)^3 - \tfrac{1}{4}(\phi-1)^4 + \text{etc.}$$
$$= \sum_{1}^{\infty}{}^{\lambda}(-1)^{\lambda+1}\frac{(\phi-1)^\lambda}{\lambda}.$$

Let $\phi^n = A_0g^n + B_0g_2^n + \ldots. + W_0g_\infty^n,$

then $(\phi-1)^\lambda = A_0(g_1-1)^\lambda + B_0(g_2-1)^\lambda + \ldots. + W_0(g_\infty-1)^\lambda$

and

$$\log\phi = \sum_{1}^{\infty}{}^{\lambda}(-1)^{\lambda+1}\left\{\frac{A_0(g_1-1)^\lambda + B_0(g_2-1)^\lambda + \ldots. + W_0(g_\infty-1)^\lambda}{\lambda}\right\}$$
$$= A_0\sum_{1}^{\infty}{}^{\lambda}\frac{(g_1-1)^\lambda}{\lambda} + \ldots. + W_0\sum_{1}^{\infty}{}^{\lambda}\frac{(g_s-1)^\lambda}{\lambda}$$
$$= A_0\log g_1 + B_0\log g_2 + \ldots. + W_0\log g_\infty.$$

Again let $\phi^n = (A_0+nA_1)g_1^n + (B_0+nB_1)g_2^n + \ldots. + (S_0+nS_1)g_s^n$

then
$$\begin{aligned}(\phi-1)^\lambda = {} & A_0(g_1-1)^\lambda + \lambda A_1g_1(g_1-1)^{\lambda-1} + B_0(g_2-1)^\lambda + \lambda B_1g_2(g_2-1)^{\lambda-1} \\ & + \ldots. + S_0(g_s-1)^\lambda + \lambda S_1g_2(g_2-1)^{\lambda-1},\end{aligned}$$

and
$$\begin{aligned}\log\phi = {} & A_0\log g_1 + B_0\log g_2 + \ldots. + S_0\log g_s \\ & + A_1 \quad + B_1 \qquad + \ldots. + S_1.\end{aligned}$$

Having defined e^ϕ and $\log\phi$ independently, let us see what $e^{\log\phi}$ becomes, using the definition given for $\log\phi$.

We have

$$\begin{aligned}e^{\log\phi} &= e^{(A_0\log g_1 + B_0\log g_2 + \ldots. + W_0\log g_\infty)} \\ &= A_0e^{\log g_1} + B_0e^{\log g_2} + \ldots. + W_0e^{\log g_\infty} \\ &= A_0g_1 + B_0g_2 \qquad + \ldots. + W_0g_\infty \\ &= \phi,\end{aligned}$$

which shows that having defined e^ϕ by the ordinary series, we were justified in defining $\log\phi$ as in (b).

(c). *Sin* ϕ.—I define $\sin\phi$ as follows:

$$\sin\phi = \frac{\phi}{1} - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \text{etc.}$$
$$= \sum_{0}^{\infty}{}^{\lambda}(-1)^\lambda\frac{\phi^{2\lambda+1}}{(2\lambda+1)!}.$$

Let $\phi^n = A_0 g_1^n + B_0 g_2^n + \dots\dots + W_0 g_\omega^n$,

then
$$\sin\phi = \sum_0^\infty{}_\lambda (-1)^\lambda \left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \dots\dots + W_0 g_\omega^{2\lambda+1}}{(2\lambda+1)!} \right\}$$
$$= A_0 \sin g_1 + B_0 \sin g_2 + \dots\dots + W_0 \sin g_\omega.$$

Again if $\phi^n = (A_0 + nA_1) g_1^n + (B_0 + nB_1) g_2^n + \dots\dots + (S_0 + nS_1) g_s^n$,

then
$$\sin\phi = A_0 \sin g_1 + B_0 \sin g_2 + \dots\dots + S_0 \sin g_s$$
$$+ g_1 A_1 \cos g_1 + g_2 B_1 \cos g_2 + \dots\dots + g_s S_1 \cos g_s.$$

(d). *Sin*$^{-1}\phi$.—I define $\sin^{-1}\phi$ as follows:

$$\sin^{-1}\phi = \frac{\phi}{1} + \frac{1}{2}\cdot\frac{\phi^3}{3} + \frac{1.3}{2.4}\cdot\frac{\phi^5}{5} + \dots\dots + \frac{1.3\dots\dots\overline{2r-1}}{2.4\dots\dots 2r}\cdot\frac{x^{2r+1}}{2r+1} + \dots\dots$$
$$= \sum_0^\infty{}_\lambda \frac{1.3\dots\dots\overline{2\lambda-1}}{2.4\dots\dots 2\lambda}\cdot\frac{\phi^{2\lambda+1}}{2\lambda+1}.$$

Let $\phi^n = A_0 g_1^n + B_0 g_2^n + \dots\dots + W_0 g_\omega^n$,

then
$$\sin^{-1}\phi = \sum_0^\infty{}_\lambda \frac{1.3\dots\dots\overline{2\lambda-1}}{2.4\dots\dots 2\lambda}\cdot\left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \dots\dots + W_0 g_\omega^{2\lambda+1}}{2\lambda+1} \right\}.$$
$$= A_0 \sin^{-1} g_1 + B_0 \sin^{-1} g_2 + \dots\dots + W_0 \sin^{-1} g_\omega.$$

Having defined $\sin\phi$ and $\sin^{-1}\phi$ independently, let us see what $\sin\sin^{-1}\phi$ becomes, using the definition given for $\sin^{-1}\phi$.

We have
$$\sin\sin^{-1}\phi = \sin \sum_0^\infty{}_\lambda \frac{1.3\dots\dots\overline{2\lambda-1}}{2.4\dots\dots 2\lambda}\cdot\frac{\phi^{2\lambda+1}}{2\lambda+1}$$
$$= \sin \sum_1^\infty{}_\lambda \frac{1.3\dots\dots\overline{2\lambda-1}}{2.4\dots\dots 2\lambda}\cdot\left\{ \frac{A_0 g_1^{2\lambda+1} + B_0 g_2^{2\lambda+1} + \dots\dots + W_0 g_\omega^{2\lambda+1}}{2\lambda+1} \right\}$$
$$= \sin\{A_0 \sin^{-1} g_1 + B_0 \sin^{-1} g_2 + \dots\dots + W_0 \sin^{-1} g_\omega\}$$
$$= A_0 g_1 + B_0 g_2 + \dots\dots + W_0 g_\omega$$
$$= \phi,$$

which shows that having defined $\sin\phi$ by the ordinary series we were justified in defining $\sin^{-1}\phi$ as in (d).

WORCESTER, MASS., *April* 15, 1892.

Simple Groups from Order 201 to Order 500.

BY F. N. COLE, PH. D.

In a recent number of the Mathematische Annalen* Dr. Otto Hölder has determined all the simple groups as far as order 200. The importance of this class of groups and the lack of any general method for their construction render it desirable to extend this census as far as possible. Only the simple groups furnish new algebraic problems, the reduction of a compound group being effected by means of a series of simple groups of lower orders. The number of simple groups is very limited. Beside the (necessarily cyclical) groups of prime order, which are always simple, Dr. Hölder finds below order 201 only two other cases, both of which have long been known, viz., a simple group of order 60, isomorphic with the alternating group of permutations of 5 letters, and a simple group of order 168, isomorphic with the group identified with the transformation of the 7th order of the elliptic modular functions.

From order 201 to order 500 there is one known simple group of order 360, isomorphic with the alternating group of permutations of 6 letters. Beside this I find only two other possibilities of compound order, both of which present more serious difficulties than I have as yet been able to meet. I do not here prove or disprove the existence 1) of a second simple group of order 360 not isomorphic with the group of even permutations of 6 letters, or 2) of a simple group of order 432. It seems, however, a measurable advance to have restricted the necessity of further consideration to these two cases.

I.—*Preliminary Theorems.*

The general method here employed is identical with that of Dr. Hölder's article, to which and to the works of Netto and Serret the reader is referred for further particulars. It is to be especially noted that, the simple or compound character being a property of a group in itself, independent of the particular form in which its operations are expressed, we deal here primarily with groups in the abstract, only recurring when convenient to their representation in terms of substitutions of n letters.

---

*Math. Ann. XL, 1, p. 55.

In the analysis of the structure of a group the theorems of Sylow* are indispensable:

I.—*A group the order of which is a power of a prime number is compound.*

II.—*If* p^a *is the highest power of a prime number* p *which divides the order* r *of a group* G, *then the subgroups of order* p^a *contained in* G *are all conjugate, and their number is* $xp+1$, *where*

$$r = p^a . \nu . (xp+1),$$

the integers ν *and* x *remaining to be determined in each particular case.*

From these Dr. Hölder deduces the two following:

III.—*If the order of a group is a product of two or of three prime factors, including the case where any of the factors are equal, the group is compound.*

IV.—*If the group* G *of Theorem II is simple, it is holoedrically isomorphic with a transitive group of substitutions of* $xp+1$ *letters.*

It follows at once from the last Theorem that the order r of G must be a divisor of $(xp+1)!$ Also, since there is only one group of substitutions of 6 letters, the alternating group of order 360, the order of which lies between 201 and 500, we must always have $xp+1>6$, except when $r=360$.

The following well known theorem† is also of great use for the present purpose:

V.—*If a transitive group* G *of substitutions of* n *letters contains a transitive subgroup of lower degree, then* G *is either doubly transitive or non-primitive.*

It is to be noted further that if a simple group G of an order above 120 is non-primitive, its number of systems of non-primitivity must exceed 6. Otherwise it would be possible‡ to construct a simple group of substitutions holoedrically isomorphic with G and affecting not more than 6 letters.

Finally, a simple group G, expressed in substitutions, cannot contain any odd substitution. For the even substitutions would then give a self-conjugate subgroup.||

---

* L. Sylow: Math. Ann., V, p. 284.

† Cf. Netto: Theory of Substitutions, p. 95, Corollary.

‡ Ibid., p. 76.

|| Ibid., pp. 85 and 81.

II.—*First Reduction of the Number of Possible Orders.*

Between 201 and 500 I find the following 84 numbers which are not powers of a single prime or products of two or of three primes:

$* 204 = 2^2 . 3 . 17$
$* 208 = 2^4 . 13$
$210 = 2 . 3 . 5 . 7$
$** 216 = 2^3 . 3^3$
$* 220 = 2^2 . 5 . 11$
$\dagger 224 = 2^5 . 7$
$* 225 = 3^2 . 5^2$
$* 228 = 2^2 . 3 . 19$
$* 232 = 2^3 . 29$
$* 234 = 2 . 3^2 . 13$
$240 = 2^4 . 3 . 5$
$* 248 = 2^3 . 31$
$* 250 = 2 . 5^3$
$252 = 2^2 . 3^2 . 7$
$* 260 = 2^2 . 5 . 13$
$264 = 2^3 . 3 . 11$
$** 270 = 2 . 3^3 . 5$
$* 272 = 2^4 . 17$
$* 276 = 2^2 . 3 . 23$
$280 = 2^3 . 5 . 7$
$288 = 2^5 . 3^2$
$* 294 = 2 . 3 . 7^2$
$* 296 = 2^3 . 37$
$* 297 = 3^3 . 11$
$** 300 = 2^2 . 3 . 5^2$
$* 304 = 2^4 . 19$
$306 = 2 . 3^2 . 17$
$* 308 = 2^2 . 7 . 11$

$* 312 = 2^3 . 3 . 13$
$315 = 3^2 . 5 . 7$
$** 320 = 2^5 . 5$
$** 324 = 2^2 . 3^4$
$* 328 = 2^3 . 41$
$* 330 = 2 . 3 . 5 . 11$
$336 = 2^4 . 3 . 7$
$* 340 = 2^2 . 5 . 17$
$* 342 = 2 . 3^2 . 19$
$* 344 = 2^3 . 43$
$* 348 = 2^2 . 3 . 29$
$* 350 = 2 . 5^2 , 7$
$351 = 3^3 . 13$
$* 352 = 2^5 . 11$
$360 = 2^3 . 3^2 . 5$
$* 364 = 2^2 . 7 . 13$
$* 368 = 2^4 . 23$
$* 372 = 2^2 . 3 . 31$
$* 375 = 3 . 5^3$
$* 376 = 2^3 . 47$
$* 378 = 2 . 3^3 . 7$
$380 = 2^2 . 5 . 19$
$** 384 = 2^7 . 3$
$* 390 = 2 . 3 . 5 . 13$
$\dagger 392 = 2^3 . 7^2$
$396 = 2^2 . 3^2 . 11$
$400 = 2^4 . 5^2$
$* 405 = 3^4 . 5$

$* 408 = 2^3 . 3 . 17$
$* 414 = 2 . 3^2 . 23$
$* 416 = 2^5 . 13$
$420 = 2^2 . 3 . 5 . 7$
$* 424 = 2^3 . 53$
$432 = 2^4 . 3^3$
$* 440 = 2^3 . 5 . 11$
$* 441 = 3^2 . 7^2$
$* 444 = 2^2 . 3 . 37$
$\dagger 448 = 2^6 . 7$
$** 450 = 2 . 3^2 . 5^2$
$* 456 = 2^3 . 3 . 19$
$* 459 = 3^3 . 17$
$* 460 = 2^2 . 5 . 23$
$* 462 = 2 . 3 . 7 . 11$
$* 464 = 2^4 . 29$
$* 468 = 2^2 . 3^2 . 13$
$* 472 = 2^3 . 59$
$* 476 = 2^2 . 7 . 17$
$480 = 2^5 . 3 . 5$
$* 484 = 2^2 . 11^2$
$* 486 = 2 . 3^5$
$* 488 = 2^3 . 61$
$* 490 = 2 . 5 . 7^2$
$* 492 = 2^2 . 3 . 41$
$495 = 3^2 . 5 . 11$
$* 496 = 2^4 . 31$
$* 500 = 2^2 . 5^3$

Of these those marked $*$ or $**$ are to be rejected, the former because for one or another factor p the number $xp+1$ can only be 1, the latter because $xp+1$ is less than 7. Further, the three cases 224, 392, and 448 are excluded, since these orders are not divisors of corresponding numbers $(xp+1)!$. The possible simple groups are therefore included among the following 18 orders:

$210 = 2.3.5.7$
$240 = 2^4.3.5$
$252 = 2^2.3^2.7$
$264 = 2^3.3.11$
$280 = 2^3.5.7$
$288 = 2^5.3^2$
$306 = 2.3^2.17$
$315 = 3^2.5.7$
$336 = 2^4.3.7$
$351 = 3^3.13$
$360 = 2^3.3^2.5$
$380 = 2^2.5.19$
$396 = 2^2.3^2.11$
$400 = 2^4.5^2$
$420 = 2^2.3.5.7$
$432 = 2^4.3^3$
$480 = 2^5.3.5$
$495 = 3^2.5.11$

III.—*Further Reduction of the* 18 *Cases to* 7.

1. Of the remaining 18 orders the following 7 are at once disposed of:

240, 280, 306, 351, 380, 396, 495.

Thus, if $r = 240 = 5.\nu.(5x + 1)$, we must take $5x + 1 = 6$ or 16. 6 is excluded. A simple group of order 240 must therefore be isomorphic with a transitive group of 240 substitutions of 16 letters. The subgroup which leaves one of the 16 letters unchanged is then of order $\frac{240}{16} = 15$. But a group of order 15 is necessarily cyclical; its operations are all powers of a single one among them, s_1. The powers of s_1, the exponents of which are divisible by 3, form a self-conjugate subgroup of order 5. Every operation of this subgroup, except identity, must consist of exactly 3 cycles of five letters each. Otherwise the same subgroup of order 5 would occur in several of the groups which leave a single one of the 16 letters unchanged, and there would therefore be less than 16 subgroups of order 5. The operation s_1 accordingly cannot consist of cycles of 5 letters with cycles of 3 letters, but must be a single cycle of 15 letters. Every power of s_1 therefore affects all the 15 letters. The group of order 240 then contains $16.14 = 224$ distinct operations of order 15, leaving only 16 further operations. There can therefore be only *one* subgroup of order 16. Consequently a group of order 240 is compound.

Again, if $r = 280 = 7.\nu.(7x + 1)$, we have $7x + 1 = 8$. There is therefore to be a simple transitive group of 280 substitutions of 8 letters. This requires a subgroup of order 35 affecting 7 letters. But a group of order 35 is cyclical, and such a group cannot be constructed with 7 letters.

For $r = 306 = 17.\nu.(17x + 1)$ we have $17x + 1 = 18$, and a simple transitive group of 306 substitutions of 18 letters, with 18 subgroups of order 17 affecting 17 letters. These give $18.16 = 288$ operations of order 17, leaving only 18 further operations, each of which must affect all the 18 letters. Among these there must be a substitution of order 2, which must then consist of 9 cycles of two letters each. But such a substitution is odd.

For $r = 351 = 13.\nu.(13x + 1)$ there must be 27 subgroups of order 13, involving $27.12 = 324$ distinct operations. There remain only 27 further operations, which can furnish only *one* group of order 27.

For $r = 380 = 19.\nu.(19x + 1)$ we have 20 subgroups of order 19, with $20.18 = 360$ distinct operations. Also from $380 = 5.\nu.(5x + 1)$ it follows that there are 76 subgroups of order 5, with 76.4 further operations. But these cannot all be included in a group of order 380.

If $r = 396 = 11.\nu.(11x + 1)$, there must be a simple transitive group of 396 substitutions of 12 letters, containing 12 subgroups of order 33 affecting 11 letters. But a group of order 33 is cyclical, and is impossible with only 11 letters.

Finally, if $r = 495 = 11.\nu.(11x + 1)$, we have 45 subgroups of order 11, containing $45.10 = 450$ distinct operations of order 11. But from $495 = 5.\nu.(5x + 1)$ it appears that there are also 11 subgroups of order 5, containing $11.4 = 44$ operations of order 5. These two sets of operations together with identity make up the entire group, leaving no opportunity for any operations of order 3.

The 7 orders just discussed therefore afford no simple groups.

2. Of the 11 cases remaining, the following 4 also present no considerable difficulty:

210, 264, 315, 336.

For $r = 210 = 7.\nu.(7x + 1)$ there must be 15 subgroups of order 7, containing $15.6 = 90$ distinct operations of this order. Also from $210 = 3.\nu.(3x + 1)$ we have $3x + 1 = 7$, 10, or 70. In the last case there would be $70.2 = 140$ operations of order 3, which could not all be included in the group. Also if $3x + 1 = 7$, there must be a simple transitive group of 210 substitutions of 7 letters. This would contain subgroups of order 30, affecting 6 letters. But a group of order 30 contains a cyclical subgroup of order 15, and this cannot be constructed with 6 letters. Finally, if $3x + 1 = 10$, the isomorphic group of 210 substitutions of 10 letters would include a circular substitution of 7 letters, and must therefore be doubly transitive or non-primitive. Both of these possibilities are here excluded. There is therefore no simple group of this order.

Again, if $r = 264 = 11 . \nu . (11x + 1)$, we require a simple, transitive group of 264 substitutions of 12 letters. This will include 12 subgroups of order 22, affecting 11 letters. Such a subgroup consists of the powers of a circular substitution of 11 letters and 11 substitutions with 5 cycles of two letters each. The latter substitutions being odd, there is no simple group of this order.

If $r = 315 = 9 . \nu . (3x + 1)$, there must be a simple, transitive group of 315 substitutions of 7 letters. This group must contain a circular substitution of 5 letters. But the group is neither doubly transitive nor non-primitive.

For $r = 336 = 7 . \nu . (7x + 1)$, we must have $7x + 1 = 8$. The corresponding simple, transitive group of 8 letters contains subgroups of 42 substitutions of 7 letters. Such a subgroup contains a further, self-conjugate subgroup of order 7, consisting of the powers of a circular substitution of 7 letters. It contains further an operation of order 2. This operation cannot consist of 1 or of 3 cycles of two letters, since both of these cases are odd. But 2 cycles of two letters cannot transform the circular substitution of 7 letters into any of its powers, as must here be the case.

IV.—*The Orders* 252, 288, 400, 420, *and* 480.

These require a more detailed consideration, involving a somewhat elaborate determination of the possible orders of the included operations.

1. $r = 252$. This case requires 36 subgroups of order 7, furnishing at once $36 . 6 = 216$ of the 252 operations. The corresponding simple, transitive group of 36 letters contains 36 subgroups of order 7 affecting 35 letters each. Every actual substitution of these subgroups must consist of 5 cycles of 7 letters each; otherwise there would not be 36 *distinct* subgroups of this type. The remaining 35 actual substitutions affect all the 36 letters. Every one of them must be regular, i. e. must consist of cycles all of which contain the same number of letters. Otherwise a proper power of one of them would affect less than 35 letters, without being identity.

There can be no operation of order 9 in the group. For such an operation must consist of 4 cycles of 9 letters each. An operation of order 7 could not transform this operation into itself or into any of its powers. Every operation of order 9 would therefore give rise, on being transformed by an operation of order 7, to 7 conjugate operations of order 9. The 1st, 2d, 4th, 5th, 7th, and

8th powers of these 7 operations must all be distinct. We should have therefore $7.6 = 42$ distinct new operations, whereas there can be only 36. There is therefore no operation of order 9, 18, or 36 in the group.

The possible orders beside 7, which must all be divisors of 36, are therefore reduced to 12, 6, 4, 3, and 2. An operation of order 12 must consist of 3 cycles of 12 letters each, and one of order 4 of 9 cycles of 4 letters each. Both of these cases are odd. There therefore remain only the orders 2, 3, and 6. The substitutions of these orders, like the hypothetical substitutions of order 9, must be joined in conjugate sets of 7 each. At least one subgroup of order 2 and one of order 3 are present. These give rise to at least 7 operations of order 2 and $7.2 = 14$ of order 3. If any operation of order 6 is present this gives rise to 7 conjugate groups of this order, the first and fifth powers of whose generating substitutions are all distinct. These would furnish the $7.2 = 14$ missing operations.

On the other hand, if no operation of order 6 is present, then no operation of order 3 can transform an operation of order 2 into itself. For in this case the product of the two operations would be of order 6. Consequently the number of the operations of order 2 is a multiple of 3 as well as of 7, and is therefore a multiple of 21. There are therefore at least 21 operations of order 2 in the group. These, with the 14 necessarily present operations of order 3 and identity, make up the 36 operations to be supplied.

In any case, then, the group contains exactly 14 operations of order 3. From these 14 operations at least 7 subgroups of order 9 are to be constructed. These groups consist of 8 commutative operations of order 3 and identity. Two such groups, if distinct, cannot have more than 3 operations in common. In the present case they must have 3 operations in common; otherwise they would furnish 16 operations of order 3. If they have 3 operations in common they furnish together all the 14 possible operations of order 3. A third group of order 9 must therefore have all its operations in common with these two. But it can have only 3 (including identity) in common with each, or 5 with both. Accordingly, from the 14 operations of order 3 only two (if any) groups of order 9 can be constructed. A group of order 252 is therefore simple.

2. $r = 288$. We must have here 9 conjugate subgroups of order 32. There is therefore to be a simple, transitive group of 288 substitutions of 9 letters.

The 9 subgroups which affect 8 letters are those of order 32. The operations of such a group are of orders 32, 16, 8, 4, and 2; 32 and 16 are here excluded. Also a circular substitution of 8 letters, being odd, is inadmissible. Every operation of the subgroups of order 32 therefore consists of cycles of 4 and of 2 letters; 1 cycle of 4 letters, or 1 or 3 cycles of two letters are odd; 2 cycles of 2 letters cannot occur in a primitive group of degree 9; 1 cycle of 4 letters with 1 of 2 letters is excluded, since its square would have 2 cycles of 2 letters. There remain only substitutions of 2 cycles of 4 letters, or 4 cycles of 2 letters. Every substitution of the 9 subgroups of order 32 therefore affects all the 8 letters. These subgroups accordingly furnish $9.31 = 279$ distinct substitutions, leaving only 9. The latter can form at the most *one* subgroup of order 9.

3. $r = 400$. In this case there must be 16 conjugate subgroups of order 25, and therefore a simple, transitive group of 400 substitutions of 16 letters. The subgroups which leave one letter unchanged are the 16 subgroups of order 25. These cannot be cyclical, since they affect only 15 letters. Any one of them therefore consists of 24 commutative substitutions of order 5 and the identical substitution. Suppose that s_1 and s_2 are any two of the substitutions of such a group. Then s_2 transforms s_1 into itself, and since s_1 contains at the most 3 cycles, s_2 must transform every cycle of s_1 into itself. Consequently the cycles of s_2 are powers of cycles of s_1 so far as s_1 and s_2 have letters in common. It follows that the 15 letters affected by the group divide into 3 transitive systems of 5 letters each, and that every substitution of the group is a combination of one, two, or three circular substitutions u, v, w, of order 5, each of which affects one of the three systems of transitivity.

No substitution of the group of order 400 can consist of a single cycle of 5 letters. For the group would then be non-primitive. The only admissible distribution of the 16 letters would be into 8 systems of 2 letters each. But 400 is not a divisor of $2^8 . 8!$ Also the substitutions of the subgroup of order 25 cannot all consist of three cycles each. For then these subgroups would furnish $16.24 = 384$ distinct operations, leaving only 16, from which only *one* subgroup of order 16 could be formed.

A subgroup of order 25 therefore contains some operations with two cycles of 5 letters. Suppose one of these to be uv. There can then be no other operation $u^i v^j (i \neq j)$ in the group, for $(uv)^{-i} u^i v^j$ would consist of a single cycle.

The 5 powers of uv cannot make up the entire subgroup. We must therefore add to these some uw, or vw, or $uv^\lambda w^\mu$. Whatever choice is made, there is essentially only one result. The subgroup must contain the combinations

$$\begin{gathered} uv, \quad uw, \quad vw^4, \\ uv^2w^4, \; uv^3w^3, \; uv^4w^2. \end{gathered}$$

These with their powers furnish all the $6.4+1=25$ operations of the subgroup.

The group of order 400 contains therefore $\frac{16}{6}.12=32$ substitutions of order 5 with two cycles. These divide into 8 groups. There must therefore be a simple, transitive group of 400 substitutions of, at the most, 8 letters. But such a group is impossible, since 400 is not a divisor of 8 !.

4. $r=420$. Here $7x+1$ must equal 15. We require therefore a simple, transitive group of 420 substitutions of 15 letters. A subgroup which leaves one letter unchanged is of order 28, and therefore contains an operation of order 7. This cannot be a circular substitution; it consists then of two cycles of 7 letters each. The subgroup composed of the powers of this operation is self-conjugate in respect to the group of order 28. Of the remaining 21 operations of this group none can be of order 14 or 28. For a cycle of 14 letters is odd, and any combination of a cycle of 7 letters with one of 2 or 4 letters contains among its powers a single cycle of 7 letters.

All the 21 remaining operations are therefore of order 2 or order 4. One of order 4 is inadmissible, for an operation of order 4 cannot transform an operation of order 7 into any of its powers except the first. And in the latter case the product of the two operations would be of order 28. Also the operations of order 2 cannot transform the operation of order 7 into itself, for then we must have operations of order 14. Consequently every operation σ of order 2 must transform the operation s of order 7 into its 6th power; σ cannot interchange the two cycles of s, for then σ must consist of 7 cycles of two letters. Accordingly σ leaves 1 letter of each cycle of s unchanged, and interchanges the remaining 6 letters of each cycle in pairs. As soon as the two fixed letters are assigned, σ is fully determined. Moreover, a given letter of the one cycle cannot be paired as fixed letter with two different letters of the other cycle. For then the product of the two corresponding substitutions of order 2 would be different from identity, but would not affect the letters of the first cycle of s, and could not therefore transform this cycle into its 6th power. It is therefore possible to construct only 7 of the 21 required substitutions of order 2.

There is then no simple group of order 420.

5. $r = 480$. We must have here either 16 or 96 subgroups of order 5. In the former case the corresponding group of substitutions of 16 letters contains subgroups of order 30, affecting 15 letters. Each of these subgroups contains a self-conjugate subgroup of order 5. Every actual operation of the latter consists of 3 cycles of 5 letters each; otherwise the group of order 480 would not contain 16 subgroups of order 5. A group of order 30 contains, moreover, a cyclical subgroup of order 15, the generating operation of which must here consist of a single cycle of 15 letters, since its 3d power is of order 5. The remaining substitutions of the group of order 30 are of order 2, and each consists of 7 cycles of 2 letters each, and is therefore odd. Consequently a simple group of order 480 contains 96 subgroups of order 5. These furnish $96.4 = 384$ distinct operations of order 5, leaving $96 - 1$ to be identified.

The number of subgroups of order 3 cannot now exceed 48. Below this limit we find $3x + 1 = 10$, 16, or 40. The last two numbers are impossible. For 40 subgroups of order 3 would furnish 80 operations, leaving only 16 from which to form groups of order 32. And 16 groups of order 3 would require a simple, transitive group of 480 substitutions of 16 letters, which, we have just shown, does not exist. There are therefore exactly 10 subgroups of order 3. The corresponding group of substitutions of 10 letters contains 10 conjugate subgroups of order 48, affecting 9 letters. Each of these contains again 1 or 3 subgroups of order 16. We show that every actual substitution of such a subgroup affects exactly 8 letters.

In the first place there can obviously be no operation of order 16 present. Also an operation of order 8 must here consist of a single cycle of 8 letters, which, being odd, is also excluded. Every operation of the groups of order 16 under consideration must therefore be of order 4 or order 2; 1 or 3 cycles of 2 letters, 1 cycle of 4 letters, and one cycle of 4 with 2 of 2 are excluded, being odd; 2 cycles of 2 letters are also excluded. There remain only 2 cycles of 4 letters, 4 cycles of 2 letters, and 1 cycle of 4 letters with one of 2 letters. The last case is also excluded, since the square of such a substitution would consist of 2 cycles of 2 letters. Every substitution of the groups of order 16 therefore affects 8 letters. The 8 letters are the same for every substitution of the same group. For if we multiply 4 cycles of 2 letters into 4 cycles of 2 letters having 7 letters in common with the first set, it is impossible to remove one letter, without removing two.

Now a primitive group of substitutions of n letters contains a subgroup actually affecting any $n - 1$ letters.* If a subgroup of order 16 contained in a group of order 48 is transformed by a proper operation of the group affecting all the 9 corresponding letters, the result is a new subgroup of order 16 affecting a different 8 of the 9 letters. Each group of order 48 therefore contains more than 1 subgroup of order 16, and therefore contains 3 such subgroups. Each of these subgroups, leaving 2 of the 10 letters unchanged, occurs in two of the groups of order 48. There are therefore $\frac{10}{2} . 3 = 15$ subgroups of order 16. The actual operations of these groups must all be different, since each group affects a different set of 8 letters. They furnish therefore $15 . 15 = 225$ new operations, whereas only 75 are admissible.

There is, then, no simple group of order 480, and the possible orders of simple groups of compound order between 201 and 500 are reduced to 360 and 432.

ANN ARBOR, *June*, 1892.

---

*Netto: p. 94, Theorem XXII.

Extrait d'une lettre de M. d'Ocagne à M. Craig.

.... Une faute s'étant glissée dans mon Mémoire *Sur certaines courbes ....*, paru en 1888 dans l'*American Journal of Mathematics*, je vous serais obligé de faire savoir à vos lecteurs qu'il y a lieu de substituer au passage qui commence à la page 59 (14e ligne en remontant) par les mots: "En particulier pour $p = 2, \dots$" et qui finit à la page 60 (6e ligne) par ceux-ci: ".... *tangente en O à la parabole*", le suivant:

"L'équation de la droite n est, d'après ce qui vient d'être vu,

$$(p+q)y - (pq-1)x - a = 0.$$

Celle du cercle décrit sur OP comme diamètre est

$$x^2 + y^2 - ax = 0.$$

Multiplions la première de ces équations par x et retranchons les l'une de l'autre; il vient

$$y^2 + pqx^2 - (p+q)xy = 0$$

ou

$$(y - px)(y - qx) = 0.$$

Ainsi, *la droite n passe par les points de rencontre du cercle OP avec les droites* $y - px = 0$ et $y - qx = 0$.

Faisons $p = 2\mu$, $q = \mu$. Nous avons alors pour équation de la courbe, en posant $c^{\frac{1}{\mu}} = \lambda$,

$$\frac{(y - 2\mu x)^2}{y - \mu x} = \lambda.$$

C'est l'équation générale des paraboles passant en O et coupant normalement l'axe Ox au second point où elles le rencontrent. Pour une telle parabole, $y - 2\mu x = 0$ est le diamètre passant au point O, $y - \mu x = 0$, la tangente en ce point. Comme d'ailleurs le point P est pris d'une manière quelconque sur Ox, on peut, dans ce cas, énoncer comme suit le théorème précédent:

Si sur une normale à une parabole, coupant cette courbe, en dehors de son pied, au point O, on prend un point P quelconque, la droite n, qui passe par les pieds des deux autres normales qu'on peut mener du point P à la parabole, rencontre le cercle décrit sur OP comme diamètre aux points où ce cercle est coupé par la diamètre issu du point O et la tangente en ce point."